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ZEROS OF MULTIVARIABLE SYSTEMS: DEFINITIONS AND ALGORITHMS

DOUGLAS KENT LINDNER

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A number of definitions of zeros of linear time-invariant multi-variable systems have appeared recently. This work surveys selected literature on these zeros. Two questions are addressed here. First, how are zeros defined and how are these definitions interrelated. Second, how can they be calculated.

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by

Douglas Kent Lindner

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ZEROS OF MULTIVARIABLE SYSTEMS:

DEFINITIONS AND ALGORITHMS

BY

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THESIS

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Abstract

A number of definitions of zeros of linear time-invariant multivariable systems have appeared recently. This work surveys selected literature on these zeros. Two questions are addressed here. First, how are zeros defined and how are these definitions interrelated. Second, how can they be calculated.

The definitions of zeros are considered for three system representations: 1) the transfer function matrix, 2) the state space representation in the frequency domain, and 3) the state space representation in the time domain. The definitions of zeros for transfer function matrices are shown to be (mostly) equivalent. However, several different sets of zeros are defined for state space representations. The interrelationships between all of these definitions is discussed in detail.

It turns out that the calculation of zeros directly from the definitions is not always tractable. The properties of zeros, however, provide several algorithms for calculating zeros. These properties are reviewed and a brief summary of the algorithms for calculating zeros is given.

Finally, a new algorithm for the calculation of invariant zeros is introduced. It is based on the geometrical properties of linear time invariant systems. This algorithm is applicable to the most general class of systems (A,B,C,D).

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CHAPTER 1

INTRODUCTION

The definition of a zero of a scalar transfer function is well known. Indeed, the properties of zeros are very important for describing the open and closed loop behavior of dynamical systems. There is a natural interest, then, in extending this concept to linear time-invariant multivariable systems. However, it is not clear just how this should be done. The approach generally taken is to define zeros for multivariable systems so that these zeros retain some property of zeros of a scalar transfer function. As it turns out, the zeros so defined also have other properties which can be considered generalizations from the scalar case.

Zeros defined for multivariable systems have been of considerable interest recently. Macfarlane, et al, have developed the theory extending the classical Nyquist-Bode and root locus techniques to linear time-invariant multivariable systems. In these generalized techniques, the zeros have a role analogous to their role in the classical theory for single input-single output systems. See [1], for example. Zeros play a major role in the construction of minimal order inverse systems [2], the construction of reduced order models [3], decoupling theory [4], and servomechanism design [5], [6]. More recently, they have been used in relating the structure and coefficients of the quadratic weighting matrices to the resulting eigenstructure of the optimal state regulator [7] and in the stability of the optimal state regulator using high gain feedback [8]. In a more theoretical context, zeros have proven useful in describing equivalence classes of linear time-invariant systems under the action of the group of state, input,

and output space transformations, state feedback, and output injection [9]. They also appear in such diverse applications as the factorization of polynomial matrices [10], and many other areas.

The first part of this paper is devoted to a survey of the existing literature on zeros for linear time-invariant multivariable systems. The representation of a multivariable system has three forms: the transfer function matrix, state space system in the time domain, and state space system in the frequency domain. Correspondingly, the definitions of zeros has been extended to each of these representations. In what follows, these definitions are given and their interrelationship is explored. This includes the relationship between different definitions of zeros for the same system representation and the interrelationship of zeros defined for different system representations. It turns out that for reachable and observable systems, the zeros defined from each representation coincide. It is only when unreachable or unobservable modes occur that differences in definitions appear.

Once zeros are defined, it is of interest to develop algorithms for their efficient calculation. As is turns out, neither the definitions nor their elementary properties are well suited to either hand calculation or calculation on a digital computer. Therefore, several properties of zeros are explored which lead to algorithms for computing zeros. Several of these properties in some way generalize the properties of zeros of scalar transfer functions.

Using these properties, several algorithms for calculating zeros have appeared in the literature. A brief review of the algorithms is included

and their similarities and differences are discussed. Then a new algorithm is introduced based on the geometrical properties of linear time-invariant systems. The algorithm, based on a fundamental subspace, is applicable to the most general class of systems (A,B,C,D).

The organization of the chapters is as follows:

Chapter 2 contains notation and preliminary mathematical results which will be useful below. This includes a discussion of the Smith form and the Smith-McMillan form of matrices. Both of these forms are extremely important to the rest of the paper. There is also a brief discussion of the relationship between discrete and continuous systems. Both types of systems are considered in this papers. Although the theory applies equally well to both, each type of system lends certain insight into particular areas of study. This will be exploited whenever possible.

Chapter 3 discusses zeros defined for transfer function matrices. These zeros are introduced first since it turns out that they are a subset of almost every other set of zeros. Several properties of these zeros, which are a direct consequence of the basic definition, are also examined.

Chapter 4 considers zeros defined for frequency domain state space representations. As state space representations provide more information than transfer function matrices, its use has resulted in a multitude of definitions of zeros. Several definitions of zeros are analyzed and their interrelationship's are delineated. Their relationship to zeros defined for a transfer function matrix is also discussed.

Chapter 5 presents several properties of zeros which are important in the construction of algorithms to calculate zeros. These include fre-

quency domain properties which follow from the definitions in Chapter 4. Also introduced are geometrical properties of zeros. These properties essentially originate in the time domain when the state space representation is used. Although these properties are also a consequence of the definitions of Chapter 4, they can be developed independantly by a geometrical analysis of system properties.

Chapter 6 gives a brief survey of the algorithms to calculate zeros which have appeared in the literature to date. A brief outline of the theoretical basis of each algorithm is given along with a discussion of the type of zeros it calculates and of its numerical limitations.

Chapter 7 presents a new algorithm for calculating zeros based on the geometrical properties of state space systems given in Chapter 5. The algorithm actually calculates a canonic form of the system which explicitly displays fundamental subspaces closely connected to zeros. From this canonic form the zeros are easily computed. Several examples are given to illustrate its use.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2.1. Introduction

Zeros are defined for both continuous and discrete systems. In fact, it is possible to discuss zeros for both types of systems at the same time. This is because once the system is transformed into the frequency domain (by Laplace or z-transform) it assumes an algebraic representation. In this form the analysis techniques are the same for both systems. However, it is pedagogically convenient to exploit this connection as some properties of zeros lend themselves nicely to continuous time interpretation while others arise naturally in a discrete setting.

In this chapter terminology, notation, and certain preliminary results are introduced. This includes, in Section 2, the continuous and discrete systems to be discussed. The connection between these system representations in the frequency domain is discussed and certain differences relevant to this work are pointed out. Section 3 presents a brief development of the matrix techniques needed to analyze these systems, namely the Smith form and the Smith-McMillan form. These forms play a fundamental role in defining zeros. These results are directly applicable to some properties of zeros and, in some cases, the method of proof will be used in later sections.

The notation in this work is as follows:

The field of real numbers will be denoted R and the field of complex numbers by $\mathcal{C}$. Some of the results below can be generalized to arbitrary fields of real and complex numbers, however, this work will consider only the fields of real and complex numbers.

The ring of polynomials in the indeterminant s is the set of polynomials in s with finite degree and coefficients in R. Let a(s) be a polynomial. If the coefficient of the term with largest degree in a(s) is 1, the polynomial is said to be monic. Let b(s) be another polynomial. The notation "$a(s)|b(s)$" means "a(s) divides b(s)". A rational function is the ratio of two polynomials. The set of rational functions forms a field.

Matrices shall be denoted by capital letters and their elements by small letters. Let A be a $m \times n$ matrix of real numbers. The notation

$$\mathrm{diag}[a_1, a_2, \ldots, a_p]\ ,\ p = \min(m,n)$$

shall mean a matrix whose elements are given by

$$a_{ij} = \begin{cases} a_i & i = j \quad i = 1,2,\ldots,m \\ 0 & i \neq j \quad j = 1,2,\ldots,n \end{cases}$$

Note that it is not assumed that $m = n$. The transpose of A will be denoted by A^T (not A'). The rank of A is the order of the largest minor which is not identically zero.

A matrix whose elements are polynomials is called a polynomial matrix. Let N(s) be a polynomial matrix. The local rank of $N(\cdot)$ for any complex number z is the rank of N(z) (each element evaluated at $s = z$) and is denoted $\rho[N(z)]$. The normal rank of $N(\cdot)$ is defined as

$$\max_{z \in \mathbb{C}} \rho[N(z)] \triangleq \rho_n[N(\cdot)]$$

Alternatively, the normal rank of N(s) can be determined by considering the

elements as members of the ring of polynomials in s and using the definition of the rank of a matrix. Suppose that $N(s)$ is square. Then $N(s)$ is said to be unimodular if its determinant is a non-zero constant. This implies that the matrix is invertible, i.e. $N^{-1}(s)$ is again polynomial matrix.

A matrix M is said to be a common left divisor of the matrices N and D if there exists matrices $\bar{N}$ and $\bar{D}$ such that

$$N = M\bar{N}, \quad D = M\bar{D}$$

The matrices N and D are said to be right multiplies of M. A matrix L is said to be a greatest common left divisor of N and D if it is a common left divisor of N andD and L is a right multiple of every other common left divisor of N and D. Suppose that these matrices are polynomial matrices. If the greatest common left divisor L of N and D is unimodular, then N and D are said to be left coprime. Right coprime matrices are defined similarly.

For vector space analysis, Roman letters will denote matrices or the maps which they represent. Script letters denote R-vector spaces. The range space of a map B is denoted $\mathcal{R}[B] = \mathcal{B}$; the null space is denoted $\mathcal{N}[B]$. The dimension of a vector space $\mathcal{X}$ is written as $d(\mathcal{X})$.

Let A: $\mathcal{X} \to \mathcal{X}$ and take a subspace $\mathcal{V} \subset \mathcal{X}$. Define

$$A\mathcal{V} \triangleq \{y \in \mathcal{X} \mid y = Ax \text{ and } x \in \mathcal{V}\}$$

$$A^{-1}\mathcal{V} \triangleq \{x \in \mathcal{X} \mid Ax \in \mathcal{V}\}$$

The subspace $\mathcal{V}$ is said to be an invariant subspace of A if $A\mathcal{V} \subset \mathcal{V}$. In addition, let B: $\mathcal{U} \to \mathcal{X}$ and F: $\mathcal{X} \to \mathcal{U}$. Now $\mathcal{V}$ is said to be an (A,B)-invariant subspace if there exists a matrix F such that $(A + BF)\mathcal{V} \subset \mathcal{V}$; i.e. $\mathcal{V}$ is an

invariant subspace of $A + BF$.

Let $\mathcal{R}$, $\mathcal{S}$ be subspaces of $\mathcal{X}$. Define

$$\mathcal{R} + \mathcal{S} = \{x \mid r + s = x,\ r \in \mathcal{R},\ s \in \mathcal{S}\}$$

If $\mathcal{R} + \mathcal{S} = \mathcal{X}$ and $\mathcal{R} \cap \mathcal{S} = 0$, then $\mathcal{R}$ and $\mathcal{S}$ form a <u>direct sum decomposition</u> of $\mathcal{X}$ and it is denoted $\mathcal{R} \oplus \mathcal{S} = \mathcal{X}$.

Let $A: \mathcal{X} \to \mathcal{X}$ and let $\mathcal{V}$ be a subspace of $\mathcal{X}$. Call the vectors x, $y \in \mathcal{X}$ <u>equivalent mod $\mathcal{V}$</u> if $x - y \in \mathcal{V}$. Define the factor space $\mathcal{X}/\mathcal{V}$ as the set of all equivalence classes

$$\bar{x} = \{y \mid y \in \mathcal{X},\ x - y \in \mathcal{V}\},\ x \in \mathcal{V}$$

To turn $\mathcal{X}/\mathcal{V}$ into a linear vector space, define

$$\bar{x}_1 + \bar{x}_2 = \overline{x_1 + x_2},\ c\,\bar{x}_1 = \overline{cx_1}$$

for x_1, $x_2 \in \mathcal{X}$, and c a real scalar. Note that $\mathcal{X}/\mathcal{V}$ is <u>not</u> a subspace of $\mathcal{X}$. Now suppose $\mathcal{V}$ is an invariant subspace of A such that $d(\mathcal{V}) = \sigma$ and $\sigma + \rho = d(\mathcal{X})$. Then chose a subspace $\mathcal{R}$ of $\mathcal{X}$ such that $\mathcal{V} \oplus \mathcal{R} = \mathcal{X}$. Chose a basis $\{r_j:\ j = 1,\ldots,\rho\}$ for $\mathcal{R}$ and a basis $\{v_i:\ i = 1,\ldots,\sigma\}$ for $\mathcal{V}$. In the basis $\{r_1,\ldots,v_\sigma\}$ the matrix A has the form

$$A = \begin{bmatrix} A^1_{\rho\times\rho} & 0 \\ A^3_{\sigma\times\rho} & A^4_{\sigma\times\sigma} \end{bmatrix}$$

The factor space $\mathcal{X}/\mathcal{V}$ is isomorphic to $\mathcal{R}$ and $A^1_{\rho\times\rho}$ is the <u>map induced in $\mathcal{X}/\mathcal{V}$ by A</u>. The matrix $A^4_{\sigma\times\sigma}$ is called <u>the restriction of A to $\mathcal{V}$</u> and is denoted

$(A \mid \mathcal{V}) = A^4_{\sigma_x\sigma}$.

For complete details of these ideas, see [11]. For a short review of the geometric concepts, in particular the (A,B)-invariant subspace, see [12].

2.2. Representations of Linear Systems

Continuous and discrete time systems are introduced here. Their interconnection is discussed to motivate the geometric presentation below.

It will be assumed that the continuous time systems are linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (2.2,1a)$$

$$y(t) = Cx(t) + Du(t) \qquad (2.2,1b)$$

where $x \in R^n$, $u \in R^m$, $y \in R^r$ and (A,B,C,D) are real constant matrices of the appropriate dimension. The frequency response characteristics of this system can be studied by taking the Laplace transform of (2.2,1). Assuming $x(0) = 0$, this yields

$$sx(s) = Ax(s) + Bu(s) \qquad (2.2,2a)$$

$$y(s) = Cx(s) + Du(s) \qquad (2.2,2b)$$

It is convenient to represent (2.2,2) in the form

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = P(s) \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ y(s) \end{bmatrix} \qquad (2.2,3)$$

The matrix P(s) is known as the system matrix [13]. Under the assumption of zero initial conditions, it contains all of the information about the internal behavior of the system that (2.2,2) contains. If only the input-output behavior of the system is of interest, the state can be eliminated and (2.2,2) can be solved for y(s) in terms of u(s):

$$y(s) = [C(sI-A)^{-1}B + D]u(s) = G(s)u(s) \tag{2.2,4}$$

The matrix G(s) is called the <u>transfer function matrix</u>. The transfer function matrix, the system matrix, and the system (2.2,1) will be the starting points for studying zeros of continuous time systems.

Consider instead of (2.2,1), the discrete system

$$x_{i+1} = Ax_i + Bu_i \tag{2.2,5a}$$

$$y_i = Cx_i + Du_i \tag{2.2,5b}$$

where $x \in R^n$, $u \in R^m$, $y \in R^r$ and (A,B,C,D) are real constant matrices of the appropriate dimension. By using z-transforms in an analogous way to Laplace transforms, the system (2.2,5), assuming zero initial state, can be transformed to

$$zx(z) = Ax(z) + Bu(z) \tag{2.2,6a}$$

$$y(z) = Cx(z) + Du(z) \tag{2.2,6b}$$

As above, this can be written as

$$\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(z) \\ u(z) \end{bmatrix} = P(z) \begin{bmatrix} x(z) \\ u(z) \end{bmatrix} = \begin{bmatrix} 0 \\ y(z) \end{bmatrix} \tag{2.2,7}$$

In a similar fashion, the transfer function matrix of (2.2,6) is calculated as

$$y(z) = [C(zI-A)^{-1}B + D]u(z) = G(z)u(z) \qquad (2.2,8)$$

Comparing (2.2,7) and (2.2,8) to (2.2,3) and (2.2,4), respectively, it is seen that the functional form is exactly the same. Hence, any definitions of zeros based on P(s) or G(s) will apply equally well to their counterparts in discrete systems. Much of the rest of the presentation is done in terms of a transformed variable, usually written s. However, this variable should be interpreted as either the Laplace transform variable, or the z-transform variable, which ever is appropriate. In a few instances, the topic under discussion will be limited to either a continuous or discrete system. This will be pointed out in the text.

It would be desirable to extend this analogy to systems represented in the forms (2.2,1) and (2.2,5). To do this it is necessary, however, to explicitly note the distinction between a reachable state and a controllable state of a system. For the system (2.2,5), let $\phi(i;i_o,x_o)$ be the transition function from the state x_o at time i_o to the state $x = \phi(i;i_o,x_o)$ at time i. A state x is <u>controllable</u> if there exists an integer i such that

$$\phi(i;0,x) = 0 \qquad (2.2,9)$$

a state x is <u>reachable</u> if there exists an integer i such that

$$\phi(0;i,0) = x \qquad (2.2,10)$$

Corresponding definitions can be given for continuous time systems. The

set of controllable (reachable) states form a subspace in state space. For continous systems, the reachable space and the controllable space are the same. This is not true for discrete systems; however, it is true that the reachable space is always contained in the controllable space. Similar remarks hold for the observable and detectable subspaces. It will be shown that if the reachable and observable subspaces are employed in characterizing zeros, then results obtained from the system representations (2.2,1) and (2.2,5) are the same. Hence, in this work only the reachable and observable subspaces will be used.

2.3. Smith Form and Smith-McMillan Form

Two of the three representations from which zeros will be defined are matrices whose elements are either polynomials, as in the system matrix P(s) (2.2,3), or rational functions, as in the transfer function matrix G(s) (2.2,4). The analysis of the properties of P(s) or G(s) is done here via two canonic matrix forms called the Smith form (for polynomial matrices) and the Smith-McMillan form (for matrices of rational functions). This section will present a brief development of these two forms. The interested reader is referred to [11] and [13] for more details and other properties.

Consider a $n \times m$ polynomial matrix A(s) of rank r. To obtain the Smith form of A(s), define the following row operations on A(s):

1) Multiplication of any row by a non-zero constant
2) Addition to any row of any other row multiplied by an arbitrary polynomial.
3) Interchange of any two rows

In a similiar fashion, define elementay column operations.

Now select the element of A(s) which has least degree. Suppose that it is $a_{ij}(s)$. Then for any other element $a_{hk}(s)$

$$a_{hk}(s) = a_{ij}(s)q(s) + r_{hk}(s) \tag{2.3,1}$$

where q(s) is a polynomial such that the degree of $r_{hk}(s)$ is zero or the degree of $r_{hk}(s)$ is less than the degree of $a_{ij}(s)$. If $a_{ij}(s)$ doesn't divide every element of A(s), then, by using elementary row and column operations, A(s) can be replaced by a matrix $A_1(s)$ which contains an element of degree lower than $a_{ij}(s)$, say $r_{hk}(s)$. Now continue this process using $r_{hk}(s)$. After a finite number of steps ℓ, the matrix $A_\ell(s)$ will contain an element which divides every other element of $A_\ell(s)$. Again using elementary row and column operations, this lowest degree element can be placed in the upper left hand corner and the other elements in the first row and column reduced to zero. Further, this element can be chosen to be monic. Denote this matrix by B(s). Then B(s) has this form:

$$B(s) = \begin{bmatrix} b_{11}(s) & 0 \ldots 0 \\ 0 & \\ \vdots & \bar{B}(s) \\ 0 & \end{bmatrix} \tag{2.3,2}$$

where $b_{11}(s)$ is a monic polynomial which divides every element of $\bar{B}(s)$.

Now repeat the process with $\bar{B}(s)$. Eventually this procedure must terminate as A(s) has finite rank. The matrix so obtained, call it S(s), is said to be the <u>Smith form</u> of A(s) and its structure is given by

$$S(s) = \mathrm{diag}[\epsilon_1(s), \epsilon_2(s), \ldots, \epsilon_r(s), 0, \ldots, 0] \tag{2.3,3}$$

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where by construction

$$\epsilon_{i-1}(s) \mid \epsilon_i(s) \quad i=2,\ldots,r \tag{2.3,4}$$

The polynomials $\epsilon_i(s)$, $i=1,\ldots,r$ are called the <u>invariant factors</u> or <u>invariant polynomials</u> of A(s).

Each of the elementary row and column operations given above can be represented by pre- and post- multiplication of A(s) by the appropriate matrices, respectively. For instance, the interchange of two rows, the ith and jth say, is accomplished by pre-multiplication of A(s) by a permutation matrix obtained from an identity matrix which has the ith and jth rows interchanged. It should be noted that each matrix representing an elementary operation is a non-singular unimodular matrix.

It therefore follows that the matrices A(s) and S(s) are related by

$$S(s) = L(s)A(s)R(s) \tag{2.3,5}$$

The $n \times n$ matrix L(s) is a product of matrices which represent elementary row operations. Since each matrix is unimodular, L(s) is also unimodular. Similar remarks apply to R(s). Two matrices, such as A(s) and S(s), related as in (2.3,5) are called <u>equivalent matrices</u>.

Since L(s) is a unimodular matrix, its local rank for any value of s is always equal to its normal rank. The same is true of R(s). It follows that

$$\rho[S(s)] = \rho[A(s)] \tag{2.3,6}$$

for all s. The special form of S(s) makes this a very useful property.

EXAMPLE 1

Let A(s) be given by

$$A(s) = \begin{bmatrix} s+1 & (s-2)^2 \\ s+3 & (s+4)(s+2) \end{bmatrix}$$

Then the following steps lead to the Smith form of A(s).

$$\begin{bmatrix} s+1 & (s-2)^2 \\ 2 & 10s+4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5s+2 \\ s+1 & (s-2)^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5s+2 \\ 0 & -4s^2-11s+2 \end{bmatrix}$$

$$S(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^2-\frac{11}{4}s+\frac{1}{2} \end{bmatrix}$$

In terms of equation (2.3,5), the relationship is given as:

$$\begin{bmatrix} 1 & 0 \\ 0 & s^2-\frac{11}{4}s+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8}(s+1) \end{bmatrix} \begin{bmatrix} s+1 & (s-2)^2 \\ s+3 & (s+4)(s+2) \end{bmatrix} \begin{bmatrix} 1 & -(5s+2) \\ 0 & 1 \end{bmatrix}$$

□

The Smith form has many important and interesting properties. However, only the relationship between the invariant polynomials and the determental divisors will be of interest here. Let $A\begin{pmatrix} i_1,i_2,\ldots,i_k \\ j_1,j_2,\ldots,j_k \end{pmatrix}; s$ denote the k-th order minor of A(s) which is formed by deleting all rows but $i_1,i_2,\ldots,i_k$ and all columns but $j_1,j_2,\ldots,j_k$. The minors of S(s) are related to the minors of A(s) by the Binet-Cauchy formula [11, p. 9]:

$$S\begin{pmatrix} i_1,i_2,\ldots,i_p \\ j_1,j_2,\ldots,j_p \end{pmatrix}; s = $$

$$\sum_{\substack{1\le\alpha_1 \ldots \alpha_p\le n \\ 1\le\beta_1 \ldots \beta_p\le m}} L\begin{pmatrix} i_1,\ldots,i_p \\ \alpha_1,\ldots,\alpha_p \end{pmatrix}; s \quad A\begin{pmatrix} \alpha_1,\ldots,\alpha_p \\ \beta_1,\ldots,\alpha_p \end{pmatrix}; s \quad R\begin{pmatrix} \beta_1,\ldots,\beta_p \\ j_1,\ldots,j_p \end{pmatrix}; s \tag{2.3,7}$$

$$p = 1,2,\ldots,\min(n,m)$$

Here the sum is over all possible permutations. Let $D_j(s)$, $j = 1,2,\ldots,r$ denote the greatest common divisor of all j-th order minors of A(s). Define $D_o(s) \equiv 1$. For the moment, denote the greatest common divisors of the jth order minors of S(s) by $D_j^*(s)$. It follows from (2.3,7) that

$$D_j^*(s) \mid D_j(s) \quad j = 1,2,\ldots,r \tag{2.3,8}$$

(The minors of orders higher than r are zero for S(s).). However, L(s) and R(s) are invertible. Therefore, A(s) and S(s) can change places in (2.3,7) when L(s) and R(s) are replaced by the appropriate matrices. It now follows that

$$D_j(s) \mid D_j^*(s) \quad j = 1,2,\ldots,r \tag{2.3,9}$$

When the determental divisors are required to be monic, it is seen from (2.3,8) and (2.3,9) that

$$D_j(s) = D_j^*(s) \quad j = 1,2,\ldots,r \tag{2.3,10}$$

It follows from the special form of S(s) that

$$D_1(s) = \epsilon_1(s),\ D_2(s) = \epsilon_1(s)\epsilon_2(s),\ldots,D_r(s) = \epsilon_1(s)\ldots\epsilon_r(s) \tag{2.3,11}$$

Note that if S(s) is square and has full rank then

$$D_r(s) = \det[S(s)] \tag{2.3,12}$$

This implies that for a square matrix of full rank, say A(s), that

$$\det[A(s)] = \alpha\,\epsilon_1(s)\,\epsilon_2(s)\ldots\,\epsilon_r(s) \tag{2.3,13}$$

where α is a non-zero constant.

<u>Example 2</u>

Let A(s) be defined as in Example 1. By inspection, the greatest commom divisor of all single elements is 1. Hence,

$$\epsilon_1(s) = 1$$

The determinant of A(s) is easily calculated as

$$\det[A(s)] = 8s^2 + 22s - 4$$

It's monic greatest common divisor is

$$s^2 + 11/4s - \frac{1}{2} = \epsilon_2(s)$$

This agrees with the Smith form calculated in Example 1. □

The analogue of the Smith form for matrices whose elements are rational functions is called the Smith-McMillan form (or McMillan form).

Consider now an $r \times m$ matrix $G(s)$ whose elements are rational functions. Let the normal rank be k. The matrix $G(s)$ can be written as

$$G(s) = \frac{1}{d(s)} N(s) \qquad (2.3,14)$$

where $d(s)$ is the monic least common demoninator of all elements of $G(s)$ and $N(s)$ is a $r \times m$ polynomial matrix. Now compute the Smith form of $N(s)$, call it $N_s(s)$. Since $N(s)$ and $N_s(s)$ are equivalent matrices, they are related as

$$N(s) = L(s)N_s(s)R(s) \qquad (2.3,15)$$

where $L(s)$ and $R(s)$ are unimodular matrices. The matrix $G(s)$ can now be expressed as

$$G(s) = \frac{1}{d(s)} L(s)N_s(s)R(s) = L(s)M(s)R(s) \qquad (2.3,16)$$

where

$$M(s) = \frac{1}{d(s)} N_s(s) \qquad (2.3,17)$$

and all possible cancellations have been carried out in $M(s)$. The matrix $M(s)$ is then the Smith-McMillan form of $G(s)$. It has the form,

$$M(s) = \text{diag}\left[\frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\epsilon_k(s)}{\psi_k(s)}, 0, \ldots, 0\right] \qquad (2.3,18)$$

By construction, $\epsilon_i(s)$ and $\psi_i(s)$, $i = 1,\ldots,k$, are coprime. Recall that $\epsilon_{i-1}(s) | \epsilon_i(s)$, $i = 2,\ldots,k$. It follows that $\psi_i(s) | \psi_{i-1}(s)$, $i = 2,\ldots,k$. It should be noted here that although $\epsilon_i(s)$ and $\psi_i(s)$ are coprime, $\epsilon_i(s)$ and $\psi_j(s)$ may <u>not</u> be coprime for $i \neq j$.

<u>Example 3</u>

Let G(s) be given by

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{s^2+s-4}{(s+1)(s+2)} & \frac{2s^2-s-8}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{2(s-2)}{s+1} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{bmatrix}$$

$$= \frac{1}{d(s)} N(s)$$

The Smith form of N(s) is easily calculated as

$$N_s(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+2)(s-2) \\ 0 & 0 \end{bmatrix}$$

Now the Smith-McMillan form of G(s), M(s),is found to be

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s-2}{s+1} \\ 0 & 0 \end{bmatrix} \qquad \square$$

CHAPTER 3

Zeros Defined from the Transfer Function Matrix

3.1. Introduction

Several authors have defined zeros from the transfer function matrix [13][14][15]. The definition of zeros for a transfer function matrix is a generalization, in some sense, of the zeros defined for a scalar transfer function. Thus, the properties of zeros defined for the transfer function matrix will be analogous to the properties of zeros of a scalar transfer function. In particular, the zeros are certain frequencies at which a non-zero input produces an identically zero output, an analogy to the definition of zeros of a scalar transfer function.

In general, the transfer function matrix $G(s)$ can be factored into polynomial matrices such that $G(s) = N(s)D^{-1}(s)$. Then the zeros are shown to cause a loss of rank in the "numerator" of $G(s)$, $N(s)$ [14]. If $G(s)$ is used to define an "inverse system" the zeros of $G(s)$ are found to be the poles of the inverse system [14]. In a recent development [15], of generalized root-locus and Nyquist-Bode techniques for multivariable systems, the zeros of $G(s)$ are shown to play an analogous role to their part in the classical theory. Clearly, zeros for multivariable systems have properties which are analogous to properties of zeros of scalar transfer functions.

It turns out that none of their properties provide a very attractive computation method for obtaining these zeros. However, the Smith-McMillan form, which provides the theoretical basis for this investigation, does provide a discription of zeros which is somewhat more managable

computationally than the other properties.

The organization of this chapter is as follows: Section 2 provides a motivation and definition of zeros. This is accomplished by the Smith-McMillan form of G(s). This same property is again described in Section 3 from a slightly different mathematical approach. It is then shown that the same set of zeros is being described. In Section 4 the zeros of G(s) are described as the poles of an inverse system. Section 5 provides a characterization of zeros in terms of the minors of G(s). Sections 2-5 depend fundamentally on the Smith-McMillan form of G(s). Section 6, then, briefly describes the zeros of G(s) in terms of complex variable theory. (This is related to generalized Nyquist-Bode and root-low criteria for multivariable systems). Section 7 concludes with a summary of the material presented. Examples to illustrate the theory are provided throughout the chapter.

3.2. Definition of Transmission Zeros

Consider a $r \times m$ matrix of rational functions, G(s). Without loss of generality, let the normal rank of G(s) be min(m,r). The transfer function matrix G(s) has a Smith-McMillan form

$$G(s) = L(s)M(s)R(s) \qquad (3.2,1)$$

where

$$M(s) = \mathrm{diag}[{}^{\epsilon_i(s)}/_{\psi_i(s)}] \quad i = 1,\ldots,\min(m,r) \qquad (3.2,2)$$

and L(s) and R(s) are unimodular matrices. Since L(s)(R(s)) has a non-zero constant determinant, its columns (rows) are linearly independent

for all s. Let

i) $\ell_i(s)$ $i = 1,\ldots,p$ be the columns of L(s)

and

ii) $r_i(s)$ $i = 1,\ldots,p$ be the rows of R(s)

where $p = \min(m,r)$. Using this notation, G(s) can be expressed as

$$G(s) = \sum_{i=1}^{p} \ell_i(s) \frac{\epsilon_i(s)}{\psi_i(s)} r_i(s) \tag{3.2,3}$$

Since G(s) relates the transformed input and output vectors by

$$y(s) = G(s)u(s) \tag{3.2,4}$$

it follows that

$$y(s) = \sum_{i=1}^{p} \ell_i(s) \frac{\epsilon_i(s)}{\psi_i(s)} r_i(s)u(s) \tag{3.2,5}$$

The relation (3.2,5) shows explicitly the relationship between the input and output for any specific frequency. Suppose that u(s) is chosen such that

$$r_j(s)u(s) = \delta_{ij}\alpha(s) \tag{3.2,6}$$

where $\alpha(s)$ is a non-zero polynomial and δ_{ij} is the Kronecker delta. Then relation (3.2,5) shows that

i) y(s) is zero if and only if $\epsilon_i(s)$ is zero

ii) y(s) is infinite if and only if $\psi_i(s)$ is zero

This argument qualitatively describes the main idea that provides the physical motivation to the following definitions of zeros and poles.* That is to say, if a complex frequency s_1 is a root of $\epsilon_i(s)$ for some i, then transmission of that frequency through the system is blocked. If a complex frequency s_2 is a root of $\psi_i(s)$ for some i, then the output becomes infinite at that frequency. These properties are similar to properties of zeros and poles for a scalar transfer function.

Therefore, let

$$z(s) = \prod_{i=1}^{p} \epsilon_i(s) \qquad (3.2,7)$$

and

$$p(s) = \prod_{i=1}^{p} \psi_i(s) \qquad (3.2,8)$$

The polynomial z(s) will be refered to as the <u>zero polynomial</u> and p(s) will be refered to as the <u>pole polynomial</u> [17]. In general,

$$z(s) = (s-z_1)(s-z_2)\ldots(s-z_k) \qquad (3.2,9)$$

where among the complex numbers z_i, $i = 1,\ldots,k$ any of them may be repeated.

<u>Definition 1</u> [13]

The <u>transmission zeros</u> of G(s) are the complex roots of z(s).

□

*This development appears in [14].

That is to say, the transmission zeros of G(s) are the complex numbers z_i, $i = 1,\ldots,k$. The zero polynomial can be written as

$$z(s) = \prod_{i=1}^{t} (s - z_i)^{k_i}, \quad k_i > 0, \; k = \sum_{i=1}^{t} k_i \tag{3.2,10}$$

where the complex numbers z_i, $i = 1,\ldots,t$ are required to be distinct. In this case, G(s) is said to have a transmission zero of order k_i at z_i.

In the sequel, it shall be useful to talk about the poles of G(s). Although the poles of a system are not the primary topic of this paper, a definition appears here for future reference. The comments above about the multiplicities of zeros apply also to poles.

<u>Definition 2</u> [17]

The <u>poles</u> of G(s) are the complex roots of p(s). □

<u>Example 4</u>

Consider the matrix G(s) given in Example 3. It has the Smith-McMillan form

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s-2}{s+1} \\ 0 & 0 \end{bmatrix}$$

Hence,

$$z(s) = s-2$$

$$p(s) = (s+1)^2(s+2)$$

This system has a transmission zero at $s = 2$ and poles at $s = -1$ and $s = -2$. The pole at $s = -1$ has a multiplicity of 2. □

3.3. Another Characterization of Transmission Zeros

In the heuristic argument in the last section, a transmission zero of a transfer function matrix $G(s)$ was shown to be a complex frequency whose transmission through the system was blocked. This was the physical motivation for defining the transmission zeros of $G(s)$ in [14] and [15] as well. However, the mathematical approach was slightly different in those references.

Again consider a $r \times m$ matrix of rational functions $G(s)$ which has full rank. It is well known [14] that $G(s)$ can be factored (non-uniquely) as

$$G(s) = D_{\ell}^{-1}(s)N_{\ell}(s) = N_r(s)D_r^{-1}(s) \tag{3.3,1}$$

where $N_{\ell}(s)$ and $N_r(s)$ are $r \times m$ polynomial matrices, $D_r(s)$ is a $m \times m$ polynomial matrix, and $D_{\ell}(s)$ is a $r \times r$ polynomial matrix. Furthermore, $N_{\ell}(s)$ and $D_{\ell}(s)$ are left coprime and $N_r(s)$ and $D_r(s)$ are right coprime. The matrices $D_r(s)$ and $D_{\ell}(s)$ are required to be non-singular. The assumption that $G(s)$ has full normal rank implies that both $N_r(s)$ and $N_{\ell}(s)$ have full normal rank.

Consider the case where $r \geq m$. Suppose there exists a complex number z such that

$$\rho[N_{\ell}(z)] < \rho_n[N_{\ell}(\cdot)] \tag{3.3,2}$$

and that z is not a pole of G(s). Then in [14], it is shown that there exists a nonzero input vector g and an appropriate set of initial conditions such that for the input

$$u(t) = ge^{zt}U(t) \tag{3.3,3}$$ *

the response of the system is identically zero for all $t > 0$. A similar result holds when $m > r$ and there exists a complex number z (not a pole) which satisfies (3.3,2).

These observations suggest that the complex number z described above is related to the transmission zeros of G(s). To see this connection, consider the Smith-McMillan form of G(s)

$$G(s) = L(s)M(s)R(s) \tag{3.3,4}$$

where

$$M(s) = \mathrm{diag}[\epsilon_i(s)/\psi_i(s)],\ i = 1,\ldots,p = \min(m,r) \tag{3.3,5}$$

Observe that M(s) can be factored in the following way:

$$M(s) = \Gamma_\ell^{-1}(s)E_\ell(s) = E_r(s)\Gamma_r^{-1}(s) \tag{3.3,6}$$

where the $r \times m$ polynomial matrix $E_\ell(s)(E_r(s))$ is given by

$$E_\ell(s) = E_r(s) = \mathrm{diag}[\epsilon_i(s)]\ i = 1,\ldots,p \tag{3.3,7}$$ †

and the $r \times r$ polynomial matrix $\Gamma_\ell(s)$ is given by

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*U(t) is the unit step function.

†Recall that G(s) is assumed to have full rank.

$$\Gamma_\ell(s) = \mathrm{diag}[\psi_1(s),\ldots\psi_p(s),1,\ldots,1] \qquad (3.3,8)$$

If $p < m$, the 1's are placed on the diagonal to make $\Gamma_\ell(s)$ invertible. The $m \times m$ polynomial matrix $\Gamma_r(s)$ has a similar form. Using this factorization, G(s) can be expressed as

$$G(s) = L(s)\Gamma_\ell^{-1}(s)E_\ell(s)R(s) = L(s)E_r(s)\Gamma_r^{-1}(s)R(s) \qquad (3.3,9)$$

Comparision of (3.3,9) to (3.3,1) shows that (3.3,9) is just a special case of (3.3,1) where

$$D_\ell^*(s) \triangleq \Gamma_\ell(s)L^{-1}(s) \quad ; \quad N_\ell^*(s) \triangleq E_\ell(s)R(s) \qquad (3.3,10)$$

$$D_r^*(s) \triangleq R^{-1}(s)\Gamma_r(s) \quad ; \quad N_r^*(s) \triangleq L(s)E_r(s) \qquad (3.3,11)$$

Since R(s) and L(s) are unimodular matrices, if for some complex number z

$$\rho[N_\ell^*(z)] < \rho_n[N_\ell^*(\cdot)] \qquad (3.3,12)$$

this implies

$$\rho[E_\ell(z)] < \rho_n[E_\ell(\cdot)] \qquad (3.3,13)$$

Now, from the special structure of $E_\ell(s)$ (see equation (3.3,7)), it is seen that (3.3,13) is true only if z is a root of

$$z(s) = \prod_{i=1}^{p} \epsilon_i(s) \qquad (3.3,14)$$

The same argument holds for $N_r^*(s)$. Finally, in [15] it is shown that if $N_\ell(s)$ and $\overline{N}_\ell(s)$ are <u>any</u> two matrices which satisfy the conditions for factorization in (3.3,1), then N(s) and $\overline{N}_\ell(s)$ are equivalent. In particular, all matrices $N_\ell(s)$ which satisfy the factorization conditions in (3.3,1) are

equivalent to $N_{\ell}^{*}(s)$ defined in (3.3,10). By an argument similar to the one above, it is seen that

$$\rho[N_{\ell}(z)] < \rho[N_{\ell}(\cdot)] \tag{3.3,15}$$

only if z is a root of (3.3,14). Similar remarks hold for matrices $N_r(s)$ which satisfy the factorization conditions of (3.3,1). From (3.3,10) and (3.3,11) it is clear that those numbers z which satisfy (3.3,15) will also satisfy

$$\rho[N_r(z)] < \rho[N_r(\cdot)] \tag{3.3,16}$$

and conversely. Thus, the following theorem has been proved:

<u>Theorem 1</u> [14]

A complex number z is a transmission zero of G(s) if and only if

$$\rho[N_{\ell}(z)] < \rho_n[N_{\ell}(\cdot)],$$

or, equivalently,

$$\rho[N_r(z)] < \rho_n[N_r(\cdot)],$$

where $N_{\ell}(s)$ $(N_r(s))$ is obtained from any factorization of G(s) given by (3.3,1). □

Two remarks are in order. First, note that Theorem 1 does not provide a way of calculating the order of a particular zero; i.e. it is not possible to calculate the numbers k_i in equation (3.2,10). Hence, it is said that Theorem 1 does not allow for multiplicities.

Secondly, suppose that none of the transmission zeros is also a

pole of G(s). Then it is seen from equation (3.3,9) that

$$\rho[G(z)] < \rho_n[G(\cdot)] \tag{3.3,17}$$

when z is a transmission zero. However, if a pole and a transmission zero coincide*, then it is not possible to assert (3.3,17). Indeed, it is not even clear how to define the rank of G(s) as some terms of G(s) are going to infinity, as s approaches the transmission zero [18].

<u>Example 5</u>

Consider the matrix G(s) given in Example 4. There it was shown that $z = 2$ is a transmission zero of G(s) by obtaining the zero polynomial. This may be checked by Theorem 1. It is easily seen that

$$\rho_n[G(s)] = 2$$

But

$$G(2) = \begin{bmatrix} \frac{1}{12} & -\frac{1}{12} \\ \frac{2}{12} & -\frac{2}{12} \\ 0 & 0 \end{bmatrix}$$

and clearly

$$1 = \rho[G(2)] < \rho_n[G(s)] = 2$$

Hence, $z = 2$ is a transmission zero of G(s) by Theorem 1. □

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*That this is possible follows from the remarks on the Smith-McMillan form in Section 2.3 and the definition of the pole and zero polynomial.

3.4. Transmission Zeros and Inverse Systems

The proof of Theorem 1 in the last section provides another characterization of transmission zeros. Consider again (3.3,1),

$$G(s) = D_\ell^{-1}(s)N_\ell(s) = N_r(s)D_r^{-1}(s) \qquad (3.4,1)$$

where $G(s)$ is a transfer function matrix and the matrices $N_\ell(s)$ and $D_\ell(s)$ are a factorization of $G(s)$ described in Section 3.3. In the same way it is shown that any two matrices $N_\ell(s)$ and $\overline{N}_\ell(s)$ are equivalent, it can be shown [15] that all matrices $D_\ell(s)$ are also equivalent. Recall the special factorization of $G(s)$ obtained from (3.3,9):

$$G(s) = L(s)\Gamma_\ell^{-1}(s)E_\ell(s)R(s). \qquad (3.4,2)$$

That is,

$$D_\ell^*(s) = \Gamma_\ell(s)L(s)^{-1}, \; N_\ell^*(s) = E_\ell(s)R(s). \qquad (3.4,3)$$

Hence, all matrices $D_\ell(s)$ are equivalent to $D_\ell^*(s)$, i.e.

$$D_\ell(s) = U(s)D^*(s)V(s) \qquad (3.4,4)$$

where $U(s)$ and $V(s)$ are unimodular matrices. It follows that

$$\det[D_\ell(s)] = \det[U(s)\Gamma_\ell(s)L^{-1}(s)V(s)] = \alpha \prod_{i=1}^{p} \psi_i(s) \qquad (3.4,5)$$

where α is a non-zero constant. This shows that the roots of the determinant of $D_\ell(s)$ are the poles of $G(s)$ (see Definition 2). This leads to another parallel between transmission zeros defined for $G(s)$ and zeros defined for scalar transfer functions; the zeros of a system are the poles

of the inverse system.

To make this precise define a <u>reflexive generalized inverse</u> [19] of a matrix X as the matrix $X^{\ddagger}$ which satisfies

$$\begin{aligned} &1) \quad X X^{\ddagger} X = X \\ &2) \quad X^{\ddagger} X X^{\ddagger} = X^{\ddagger} \end{aligned} \tag{3.4,6}$$

The reflexive generalized inverse $\overset{\ddagger}{G}(s)$ of G(s) can be obtained from (3.4,2) by inspection as

$$\overline{G}^{\ddagger}(s) = R^{-1}(s)\, \overline{E}_{\ell}(s)\, \Gamma_{\ell}(s) L^{-1}(s) \tag{3.4,7}$$

where $\overline{E}_{\ell}(s)$ is obtained from $E_{\ell}(s)$ by transposing $E_{\ell}(s)$ and inverting the non-zero elements. Using equations (3.4,2) and (3.4,7), it is easily seen that $G^{\ddagger}(s)$ satisfies (3.4,6). This gives the following theorem:

<u>Theorem 2</u> [14]

The transmission zeros of G(s) are the poles of $G^{\ddagger}(s)$ where $G^{\ddagger}(s)$ is the reflexive generalized inverse of G(s). □

<u>Example 6</u>

Consider a $r \times m$ transfer function matrix of full rank where $r \leq m$. Then there exists a right pseudo-inverse of G(s), $G^{+}(s)$ such that

$$G(s)G^{+}(s) = I_r$$

Clearly, $G^{+}(s)$ satisfies (3.4,6). Therefore, the zeros of G(s) are the poles of $G^{+}(s)$. A similar result holds for $m \leq r$. □

Example 7

Let G(s) be the transfer function matrix given in Example 5. Using equations (3.4,2) and (3.4,7), the matrix $G^{\ddagger}(s)$ is calculated as

$$G^{\ddagger}(s) = \begin{bmatrix} (s+2)(s+1) & 0 & 0 \\ \frac{(s+1)(s^2-s-8)}{s-2} & \frac{s+1}{s-2} & 0 \end{bmatrix}$$

By inspection, $M^{\ddagger}(s)$ the Smith-McMillan form of $G^{\ddagger}(s)$ is seen to be

$$M^{\ddagger}(s) = \begin{bmatrix} \frac{s+1}{s-2} & 0 & 0 \\ 0 & (s+2)(s+1) & 0 \end{bmatrix}$$

The pole of $G^{\ddagger}(s)$ is at $s=2$, which is a transmission zero of G(s) by Theorem 2. □

3.5. A Characterization of Transmission Zeros By Minors

While the transmission zeros of a transfer function matrix G(s) can be obtained from Definition 1 or Theorems 1 or 2, it is desirable to have a characterization of transmission zeros which is computationally more managable. Such a characterization can be obtained from the minors of G(s) [17].

Consider a $r \times m$ transfer function matrix G(s) with full rank. Let

$$G(s) = \frac{1}{d(s)} N(s) \tag{3.5,1}$$

where d(s) is the monic least common denominator of all elements of G(s),

and N(s) is a polynomial matrix. Let the Smith form of N(s) be $N_s(s)$ where

$$N_s(s) = \text{diag}[s_i(s)] \quad i = 1,\ldots,p; \quad p = \min(m,r) \tag{3.5,2}$$

Let the Smith-McMillan form of G(s), M(s), be given by

$$M(s) = \text{diag}\left[\frac{\epsilon_i(s)}{\psi_i(s)}\right], i = 1,\ldots,p \tag{3.5,3}$$

Recall that the $D_j(s)$ is defined to be the monic greatest common denominator of all j-square minors of N(s) where D_o is defined to be 1. As was shown in Section 2.3

$$D_1(s) = s_1(s) \tag{3.5,4}$$

It follows that

$$\frac{D_1(s)}{d(s)} = \frac{s_1(s)}{d(s)} = \frac{\epsilon_1(s)}{\psi_1(s)} \tag{3.5,5}$$

Since d(s) was specified to be the least common denominator of all elements of G(s), it follows that

$$\psi_1(s) = d(s) \tag{3.5,6}$$

That is to say, d(s) is the monic least common denominator of all 1 by 1 minors of G(s).

Similarly,

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$$\frac{D_2(s)}{d^2(s)} = \frac{s_1(s)s_2(s)}{d^2(s)} = \frac{\epsilon_1(s)\epsilon_2(s)}{\psi_1(s)\psi_2(s)} \tag{3.5,7}$$

In the right hand term, cancellations, if they occur, are between $\psi_1(s)$ and $\epsilon_2(s)$. This is because of the dividing properties of the invariant factors and the fact that $\epsilon_i(s)$ and $\psi_i(s)$ are, by definition, coprime. Let $\overline{\Psi}(s)$ be the polynomial obtained from $\psi_1(s)\psi_2(s)$ after all possible cancellations have been made. Since $D_2(s)$ is the greatest common divisor of all 2×2 minors of $N(s)$, it follows that $\overline{\Psi}(s)$ is the least common denominator of all 2×2 minors of $G(s)$. However, all of the factors in $\psi_2(s)$ appear in $\overline{\Psi}(s)$. It follows that $\psi_1(s)\psi_2(s)$ is the least common denominator of all 1×1 and 2×2 minors.

By continuing this argument in the obvious way, it is found that $p(s)$, the pole polynomial, is the least common denominator of all minors of all orders of $G(s)$. It follows similarly that the zero polynomial is the greatest common divisor of the numerators of all minors of maximum order of $G(s)$ provided these minors have been adjusted to have $p(s)$ as their denominator. These results are summarized in the following theorem:

<u>Theorem 3</u> [17]

The poles of $G(s)$ are the complex roots of the least common denominator of all non-zero minors of all orders of $G(s)$.

The transmission zeros of $G(s)$ are the roots of the greatest common divisor of the numerators of all non-zero minors of maximum order of $G(s)$ provided they have been adjusted to have $p(s)$ as their denominator. □

From the above development it is also clear that the order of a transmission zero can be identified by this characterization.

Example 8

Let G(s) be the transfer function given in Example 7. Using the notation for minors introduced in Section 2.3, the transmission zeros of G(s) can be calculated by Theorem 3 as follows:

$$G\begin{pmatrix}1,2\\ & ;s\\ 1,2\end{pmatrix} = \frac{3(s-2)}{(s+1)^2(s+2)}$$

$$G\begin{pmatrix}1,3\\ & ;s\\ 1,2\end{pmatrix} = \frac{3(s-2)}{(s+1)^2(s+2)}$$

$$G\begin{pmatrix}2,3\\ & ;s\\ 1,2\end{pmatrix} = \frac{3s(s-2)}{(s+1)^2(s+2)}$$

Now it is easily seen that

$$p(s) = (s+1)^2(s+2)$$

and

$$z(s) = (s-2)$$

which agrees with the previous results. □

3.6. Zeros Obtained From Complex Variable Theory

Macfarlane and Postlethwaite [16], in the course of deriving a generalized Nyquist stability criterion for multivariable systems, develop another characterization of zeros of a transfer function G(s). This characterization is valid only for continuous time systems. The development will only be outlined here. The interested reader is referred to [16].

Consider a $m \times m$ matrix of rational functions $G(s)$ which has full rank. For a specific value of complex frequency, say $\bar{s}$, the matrix $G(\bar{s})$ has m eigenvalues, $q_i(\bar{s})$ $i = 1,\ldots,m$. It is desired to express these eigenvalues as functions of frequency. Proceeding in an obvious way, let

$$\det[q(s)I_m - Q(s)] \stackrel{\Delta}{=} \Delta(q,s) = 0 \tag{3.6,1}$$

Let

$$\Delta(q,s) = q^m(s) + a_1(s)q^{m-1}(s) + \ldots + a_m(s) = 0 \tag{3.6,2}$$

where the $a_i(s)$ are rational functions. Let $b_o(s)$ be the least common denominator of $a_i(s)$, $i = 1,\ldots,m$. Then

$$b_o(s)\Delta(q,s) = b_o(s)q^m + b_1(s)q^{m-1} + \ldots + b_m(s) = 0 \tag{3.6,3}$$

where now the b_i, $i = 0,1,\ldots,m$ are polynomials in s. In complex analysis these functions are called algebraic functions. These functions have many interesting and important properties, and they are central to the Nyquist criterion. The properties relevant to this discussion shall simply be stated. The interested reader is refered to [16] and [20].

It can be shown that $q(s) = 0$ when $b_m(s) = 0$ and $q(s) \to \infty$ when $b_o(s) \to 0$. This suggests these definitions of poles and zeros of $G(s)$:

<u>Definition 3</u> [16]

The zeros of $G(s)$ are the complex roots of $b_m(s)$.

<u>Definition 4</u> [16]

The poles of $G(s)$ are the complex roots of $b_o(s)$.

To delineate the relationship of these zeros with the transmission

zeros defined by Definition 1, consider this expansion of the determinant $\Delta(q,s)$:

$$\det[qI_m - G(s)] = q^m - [\text{trace of } G(s)]q^{m-1} + (\Sigma \text{ principal minors of } G(s) \text{ of order } 2]\, q^{m-2} - \ldots + (-1)^m \det Q(s) = 0 \tag{3.6,4}$$

Hence, $b_o(s)$ is the least common denominator of all non-zero <u>principal</u> minors of all orders. In general, $b_o(s)$ will differ from $p(s)$ by a factor $e(s)$, i.e.

$$p(s) = e(s)b_o(s) \tag{3.6,5}$$

Since $G(s)$ is a square matrix, the zeros defined by Definition 1 will be the roots of the numerator of $\det[G(s)]$ provided the denominator is adjusted to be $p(s)$. Therefore,

$$\det[G(s)] = a_m(s) = \frac{b_m(s)}{b_o(s)} = \frac{e(s)b_m(s)}{p(s)} \tag{3.6,6}$$

These remarks show that the poles and zeros defined from the algebraic function (3.6,3) are a subset of the poles and transmission zeros defined from the Smith-McMillan form of $G(s)$. These two sets of poles and zeros differ by a set of poles, which are also zeros, that are introduced by the non-principal minors of $G(s)$.

3.7. Summary

This chapter has defined the transmission zeros of a transfer function matrix $G(s)$. The motivation for this definition is based on the

property that for certain inputs, transmission of a certain frequency though the system is blocked. Transmission zeros were then characterized by a rank test of G(s) (Theorem 1), the poles of an inverse matrix for G(s) (Theorem 2), the minors of G(s) (Theorem 3), and, for square matrices G(s), the roots of complex functions obtained from G(s) (Definition 4). If Theorems 1,2, or 3 are used to compute the transmission zeros of G(s), the zeros so obtained will be those defined by Definition 1 up to multiplicities. Definition 4, in general, defines a smaller set of zeros.

In the literature, all of the characterizations of transmission zeros contained in Theorems 1,2, and 3 have been used to define transmission zeros. It is easy to see that if any one of these three theorems is used to define transmission zeros, the characterization given by Definition 1 can be recovered as a theorem. It should also be noted that the term "transmission zero" is not standard in the literature. Some authors use "transmission zeros" to refer to a larger class of zeros defined from state space representations of system. This will be discussed in the next chapter.

CHAPTER 4

DEFINITIONS OF ZEROS BASED ON STATE-SPACE REPRESENTATIONS

4.1. Introduction

The investigation of zeros of a state space system is carried out in a manner similar to the analysis of transmission zeros when the system matrix replaces the transfer function matrix as the system representation, and the Smith form replaces the Smith-McMillan form as the mathematical tool of analysis. The motivation for defining zeros for state space systems is analogous to the motivation for transfer function matrices. It is desired that the zeros of a state space system be a set of frequencies whose transmission through the system is blocked. Several analogous results are obtained, i.e. the zeros of the system matrix are characterized in terms of the invariant polynomials, rank tests, and the minors of the system matrix.

The connection between transfer function matrices and state space systems is well known. So it is natural to ask how the transmission zeros defined for transfer function matrices are related to zeros defined for state space systems. In general, state space representations contain more information than transfer function matrices. It is not surprising then that the zeros defined for state space systems include transmission zeros as a subset, but they are, in general, a larger set. This requires that a new type of zero be introduced. It turns out that these new zeros are the eigenvalues of the unreachable and/or unobservable modes. Thus they reflect exactly the information that does not appear in the transfer function matrix.

Several definitions of zeros based on the system matrix are introduced. Each of these definitions describe a different subset of the set of transmission zeros and the eigenvalues of the unreachable and/or unobservable modes. The relationship between these sets of zeros is investigated and described as far as presently possible.

The discussion of zeros in this chapter is for state space systems represented in the frequency domain. As was discussed in Section 2.2, when the Laplace transform is applied to a continuous system (2.2.1) and a z-transform is applied to a discrete system (2.2.5), the functional forms of the resulting systems are the same. Furthermore, if the transformed system is represented by a system matrix there is no loss of information. Since the theory here applies equally well to continuous and discrete systems, state space systems will be referred to by their system matrix to introduce generality into the notation.

Section 2 provides the motivation for and definition of a set of zeros for a state space representation called invariant zeros. These zeros are then characterized by a rank test and minors of the system matrix. A slightly different definition which also appears in the literature is also discussed. The Smith form of $P(s)$ is the main analytical tool used in developing these results.

Section 3 explores the interrelationship between transmission zeros and invariant zeros. This results in the introduction of decoupling zeros. Decoupling zeros are shown to be the eigenvalues of the unreachable and/or unobservable modes of the system. The invariant zeros are then shown to contain all of the transmission zeros and some, but in general not all, decoupling zeros.

Section 4 introduces and defines system zeros. Then it is shown that the set of system zeros is the set of transmission zeros and decoupling zeros, and contains the set of invariant zeros as a subset.

An example is carried through this chapter to illustrate these definitions.

4.2. Invariant Zeros

Zeros defined for a state space representation can be characterized in a way similar to the way transmission zeros are related to frequencies whose propagation through a system is blocked. Let P(s) be a $(n+r) \times (n+m)$ system matrix. In [17] it is shown that a system excited by an input of the form

$$u(t) = g \exp(zt) U(t) \tag{4.2.1}$$

will produce an identically zero output if and only if

$$P(z) \begin{bmatrix} x_o \\ g \end{bmatrix} = 0 \tag{4.2.2}$$

where x_o is an appropriate initial state. Since x_o and g are vectors, (4.2.2) suggests that P(s) must lose rank at $s = z$. To obtain the precise relationship, the Smith form $P_s(s)$ of P(s) is used to investigate the structure of P(s). The matrix $P_s(s)$ is given by

$$P_s(s) = \mathrm{diag}[\epsilon_1(s), \epsilon_2(s), \ldots, \epsilon_p(s), 0, \ldots, 0] \tag{4.2.3}$$

where p is the normal rank of P(s) (see Section 2.3). Again, the zero polynomial z(s) is defined as

$$z(s)\epsilon = \prod_{i=1}^{p} {}_i(s). \tag{4.2.4}$$ *

The zero polynomial can be factored into linear factors

$$z(s) = (s-z_1)(s-z_2)\cdots(s-z_k) \tag{4.2.5}$$

where any one of the complex numbers z_i, $i = 1,\ldots,k$ may be repeated. This gives the following definition:

Definition 5 [17]

The invariant zeros of the system matrix P(s) are the complex numbers z_i, $i = 1,\ldots,k$; the roots of the zero polynomial $z(s)$ of $P(s)$. □

The name invariant zero comes from the fact that invariant zeros are the roots of the invariant polynomials of P(s).

Example 9 [17]

Let a system be defined by the matrices (A,B,C,D) where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad D \equiv 0.$$

The system matrix P(s) is given by

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*There is some ambiguity by also defining this polynomial as the zero polynomial. Hopefully, this will be justified below.

$$P(s) = \left[\begin{array}{cccccc:cc} s-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & s-1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & s-3 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & s+4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s+1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & s-3 & 1 & 1 \\ \hdashline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

The Smith form of P(s), $S_p(s)$ can be calculated as

$$P_s(s) = \begin{bmatrix} I_7 & 0 \\ 0 & (s+1)(s-2) \\ 0 & 0 \end{bmatrix}$$

The invariant zeros are $z = -1$ and $z = 2$. □

Definition 5 and properties of the Smith form (see (2.3,6)) yield a precise statement to the rank characterization of invariant zeros suggested by (4.2,2). This is given by the following theorem.

Theorem 4 [21]

A complex number z is an invariant zero of a system matrix P(s) if and only if

$$\rho[P(z)] < \rho_n[P(\cdot)]$$ □

The proof of this theorem was given in Section 2.3.

As an alternative approach for defining zeros for a system matrix P(s), several authors have defined these zeros directly from the rank condition suggested by (4.2,2) in the following way:

Definition 6 [22]

A complex number z is a zero of a system matrix P(s) if

$$\rho[P(z)] < n + \min(m,r) \qquad \square$$

Davison and Wang [19] call this set of zeros "transmission zeros." As will be shown below, this set of zeros is not the same as those zeros defined by Definition 1. In this paper, "transmission zero" shall refer only to a zero defined by Definition 1.

Note that if

$$\rho_n[P(\cdot)] = n + \min(r,m) \tag{4.2,6}$$

then the zeros defined by Definition 6 are the same as invariant zeros (Definition 5) as Theorem 4 shows. However, if

$$\rho_n[P(\cdot)] < n + \min(r,m) \tag{4.2,7}$$

then any complex number z will satisfy Definition 6. Hence, if (4.2,7) holds, the set of zeros obtained from Definition 6 is the entire complex plane. In general, a system whose system matrix P(s) satisfies (4.2,7) is called degenerate [22].

Example 10

Let P(s) be defined as in Example 9. Since the Smith form has rank 8 so does P(s), i.e. P(s) has full normal rank. Consider P(-1). Inspection shows that the fifth column of P(-1) is all zeros. Hence,

$$\rho[P(-1)] < \rho_n[P(\cdot)]$$

Therefore, by Theorem 4, $z = -1$ is an invariant zero. Similarly, $z = 2$ can

be checked. Note that these same two zeros would be obtained using Definition 6. □

When transmission zeros were introduced by Definition 1 the order of a transmission zero was also defined (3.2,10). Similarly, define the algebraic order of an invariant zero as follows: Let the zero polynomial z(s), defined by (4.2,4), be factored as

$$z(s) = \prod_{i=1}^{h} (s-z_i)^{k_i}, \quad k = \sum_{i=1}^{h} k_i, \quad k_i > 0 \qquad (4.2,8)$$

where the complex numbers z_i, i = 1,..., h are required to be distinct. Then z_i is said to be an invariant zero with algebraic order (or multiplicity) k_i. Note that Theorem 4 doesn't permit the algebraic order of an invariant zero to be calculated. However, for state space systems, the geometric order (or multiplicity) of an invariant zero is defined to be the rank deficiency of $P(z_i)$ [17]. Systems characterized by equality of these two multiplicities for all distinct invariant zeros are said to have simple structure. If these two multiplicities are different for any invariant zero, the systems are said to have non-simple structure [17].

Definition 5 and Theorem 4 do not provide very tractable methods for computing invariant zeros, even for systems of a modist order. This motivates the following theorem which provides an alternative method for calculating zeros. It's proof follows directly from the properties of the Smith form, Section 2.3.

Theorem 5 [17]

The invariant zeros of P(s) are the complex roots of the monic greatest common divisor of all non-zero minors of maximum order of P(s). □

Note that the algebraic order of an invariant zero can be obtained from Theorem 5. This theorem is useful when $m = r$ and the system is non-degenerate. In this case there is only one minor of maximum order so that the invariant zeros are the complex roots of

$$z(s) = \det[P(s)] = 0 \qquad (4.2,9)$$

provided that $z(s)$ has been adjusted to be monic.

The property expressed by this theorem is sometimes used to define invariant zeros. From the comments on the Smith form, Section 2.3, it is easy to see that Definition 5 would be a direct consequence of this new definition. Sometimes the zeros of $P(s)$ are defined to be the greatest common divisor of all minors of maximum order of $P(s)$ with the understanding that if the minors of maximum are all zero (the system is degenerate), the set of zeros is the whole complex plane. This is equivalent to Definition 6.

<u>Example 11</u>

Consider $P(s)$ given in Example 9. The minors of order 8, the maximum order, are calculated as:

$$P(\begin{smallmatrix}1,\ldots,8\\1,\ldots,8\end{smallmatrix};s) = -\frac{1}{2}(s+1)(s+4)(s-3)(s-2)$$

$$P(\begin{smallmatrix}1,\ldots,8\\1,\ldots,7,9\end{smallmatrix};s) = 0$$

$$P(\begin{smallmatrix}1,\ldots,8\\1,\ldots,6,8,9\end{smallmatrix};s) = 4(s-1)(s+1)(s+4)(s-2)$$

$$P(\begin{smallmatrix}1,\ldots,8\\1,\ldots,5,7,8,9\end{smallmatrix};s) = -2(s+1)(s+4)(s-2)$$

$$P(\begin{smallmatrix}1,\ldots,8\\1,..,4,6,.\,;9\end{smallmatrix};s) = 0$$

$$P(\begin{smallmatrix}1,8..,8\\1,2,3,;5,..,9\end{smallmatrix};s) = 4(s+1)(s-1)(s-2)$$

$$P\binom{1,\ldots,8}{1,2,4,\ldots,9};s) = 2(s+4)(s+1)(s-2)$$

$$P\binom{1,\ldots,8}{1,2,4,\ldots,9};s) = 2(s+4)(s+1)(s-2)$$

$$P\binom{1,\ldots,8}{1,3,\ldots,9};s) = 0$$

$$P\binom{1,\ldots,8}{2,\ldots,9};s) = 4(s+4)(s+1)(s-2)$$

The monic greatest common divisor of the non-zero minors is easily seen as

$$(s-2)(s+1)$$

Hence, the invariant zeros are $z=2$ and $z=-1$ which checks with previous calculations. □

4.3. Transmission Zeros, Decoupling Zeros, and Invariant Zeros

It is natural to ask if invariant zeros are related to transmission zeros. The answer is given, in part, by the following theorem.

Theorem 6 [13, p. 111]

For reachable and observable systems, the invariant zeros coincide with the transmission zeros. □

Theorem 6 is simply a restatement of a theorem due to Rosenbrock which states that the invariant factors of the Smith form of P(s) are exactly those polynomials which appear as the numerator polynomials of the Smith-McMillan form of G(s) provided that the state space system which gives rise to P(s) and G(s) is reachable and observable.

When the system is not reachable and/or observable, the invariant zeros are, in general, a larger class than the transmission zeros. To completely characterize all invariant zeros, a new kind of zero is introduced called a decoupling zero.

Definition 7 [13]

The input decoupling zeros are the complex roots of the invariant factors of

$$[sI-A \vdots -B] \qquad \square$$

Definition 8 [13]

The output decoupling zeros are the complex roots of the invariant factors of

$$\begin{bmatrix} sI-A \\ C \end{bmatrix} \qquad \square$$

Definition 9 [13]

The input-output decoupling zeros are those decoupling zeros that are both input decoupling zeros and output decoupling zeros. $\square$

The following theorem is immediate from the Smith form of the matrices in Definitions 7 and 8. Note that

$$\rho_n[sI-A \vdots -B] = n \qquad (4.3,1)$$

because

$$\rho_n[sI-A] = n \qquad (4.3,2)$$

for all A.

Theorem 7

The input decoupling zeros are the $z \in \mathbb{C}$ such that

$$\rho[zI-A \vdots -B] < n .$$

The output decoupling zeros are the $z \in \mathbb{C}$ such that

$$\rho \begin{bmatrix} zI-A \\ C \end{bmatrix} < n \qquad \square$$

Thus, the input decoupling zeros are the eigenvalues of the unreachable modes and the output decoupling zeros are the eigenvalues of the unobservable modes of the system.

Example 12:

Let P(s) be the system matrix in Example 9. By Theorem 7, $z = -4$ is a input decoupling zero and $z = -1$ is an output decoupling zero. □

For systems which are not reachable and/or observable the set of invariant zeros contains both transmission zeros and decoupling zeros, in general. The purpose of the rest of this section is to describe which of these zeros appear as invariant zeros.

First, it is useful to investigate why decoupling zeros don't appear among the transmission zeros. Of course, it is related to the fact that the transfer function matrix represents only the reachable and observable parts of a system. This can be made precise in the following way: Write the Smith form of the matrix in Definition 7 as

$$[sI-A \,\vdots\, -B] = L_1(s) \begin{bmatrix} \beta_1(s) & & & \\ & \ddots & & 0 \\ 0 & & \beta_n(s) & \end{bmatrix} R_1(s) \tag{4.3,3}$$

where $L_1(s)$ and $R_1(s)$ are unimodular matrices. Form the $n \times n$ matrix $\bar{Q}_1(s)$ as

$$\bar{Q}_1(s) = \mathrm{diag}[\beta_1(s),\ldots,\beta_n(s)] \tag{4.3,4}$$

and let

$$Q_1(s) = L_1(s)\bar{Q}_1(s) \tag{4.3,5}$$

Now (4.3,3) can be written as

$$[sI-A \vdots -B] = L_1(s)\bar{Q}_1(s)[I_n 0]R_1(s)$$

$$= Q_1(s)[\overline{sI-A} \vdots -\bar{B}] \qquad (4.3,6)$$

This shows that $Q_1(s)$ is a common left divisor of (sI-A) and (-B). Furthermore, $Q_1(s)$ will contain all of the input decoupling zeros. Now the matrix $Q_1(s)$ is invertable by construction as is $\overline{(sI-A)}$. Hence, when the transfer function matrix of the original system is formed, it is found that

$$G(s) = C(sI-A)^{-1}B + D = C[\overline{(sI-A)}^{-1}Q_1^{-1}(s)][Q_1(s)\bar{B}] + D$$

$$= C(\overline{sI-A})^{-1}\bar{B} + D \qquad (4.3,7)$$

This shows that the input decoupling zeros "cancel out" when the transfer function matrix is formed so that they do not appear among the transmission zeros. A similar analysis can be carried out on the output decoupling zeros. They represent the presence of a common right divisor between (sI-A) and C.

This analysis can be used to determine which transmission zeros and decoupling zeros appear as invariant zeros. Consider a system matrix P(s). Assume that $r \le m$ and, for simplicity, assume that the normal rank of P(s) is $n + r$. First, the matrix $Q_1(s)$ is obtained from (sI-A) and (-B) as above (see (4.3,3)-(4.3,6)). It contains the input decoupling zeros. Then, in a similar fashion, a matrix $Q_2(s)$ is obtained from $(\overline{sI-A})$ and C so that $Q_2(s)$ contains the remaining output decoupling zeros; that is, $Q_2(s)$ contains the output decoupling zeros minus the input-output decoupling zeros. Now P(s) can be written as

$$P(s) = \begin{bmatrix} Q_1(s) & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} \overline{(sI-A)} & -\bar{B} \\ \bar{C} & D \end{bmatrix} \begin{bmatrix} Q_2(s) & 0 \\ 0 & I_m \end{bmatrix}$$

$$= Q_1^*(s)\ \bar{P}(s)\ Q_2^*(s) \tag{4.3,8}$$

The matrix $\bar{P}(s)$ can be thought of as representing a new system, obtained from the old system, such that the new system is reachable and observable.

The minors of $P(s)$ are described using the formula (2.3,7) introduced in Section 2.3,

$$P\left(\begin{matrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{matrix}; s\right) =$$

$$\sum_{\substack{1 \le \alpha_1 < \ldots < \alpha_p \le n+r \\ 1 \le \gamma_1 < \ldots < \gamma_p \le n+r}} Q_1^*\left(\begin{matrix} i_1, \ldots, i_p \\ \alpha_1, \ldots, \alpha_p \end{matrix}; s\right) \bar{P}\left(\begin{matrix} \alpha_1, \ldots, \alpha_p \\ \gamma_1, \ldots, \gamma_p \end{matrix}; s\right) Q_2^*\left(\begin{matrix} \gamma_1, \ldots, \gamma_p \\ j_1, \ldots, j_p \end{matrix}; s\right) \tag{4.3,9}$$

By Theorem 5, the greatest common divisor $z(s)$ of all non-zero minors of maximum order determines the invariant zeros. Set $p = n + r$ to calculate these minors. Note that in this sum, every minor of maximum order of $\bar{P}(s)$ appears. From the discussion on Smith form (Section 2.3), it follows that the invariant polynomials of $\bar{P}(s)$ will appear as factors of $z(s)$. Since $\bar{P}(s)$ is reachable and observable, by Theorem 6 they represent the transmission zeros. Hence, the invariant zeros of $P(s)$ contain all of the transmission zeros of the system.

Now $Q_1^*(s)$ has only one minor of order $n + r$, so it appears as a factor in every term in the sum (4.3,9) and as a factor of $z(s)$. Since $Q_1^*(s)$ is square, its minor of order $n + r$ is its determinant. From (4.3,5) and (4.3,8), it is easily seen that

$$\det[Q_1^*(s)] = \det[Q_1(s)] = \alpha\beta_1(s)\beta_2(s)\ldots\beta_n(s) \qquad (4.3,10)$$

where α is a non-zero constant. Therefore, all of the input decoupling zeros appear as invariant zeros. If $m = r$, this same analysis holds for the output decoupling zeros. If $r < m$ this is no longer true, i.e. the invariant zeros will contain only some of the output decoupling zeros. A concise statement of these results is given in the following theorem.

Theorem 8 [23]

Given a system in which $r = m$, the set of invariant zeros consists of the set of transmission zeros, the set of input decoupling zeros, and the set of output decoupling zeros minus the input-output decoupling zeros.

If $r < m$, the set of invariant zeros consists of the set of transmission zeros, the set of input decoupling zeros, and possibly some of the set of output decoupling zeros minus the input-output decoupling zeros.

If $m < r$, the set of invariant zeros consists of the set of transmission zeros, the set of output decoupling zeros, and possibly some of the set of input decoupling zeros minus the input-output decoupling zeros. □

It would be desirable to characterize those decoupling zeros that do not appear as invariant zeros. One characterization is given in [23].

Example 13:

Consider the system matrix P(s) given in Example 9. It is shown there and in Example 12 that the invariant zeros are $z = -1$ and $z = 2$, that $z = -4$ is an input decoupling zero, and that $z = -1$ is an output decoupling zero. Note that $m = 2 < 3 = r$. In this example, the input decoupling is not an invariant zero, which is consistant with

Theorem 8. It also follows from Theorem 8 that $z = 2$ is a transmission zero. This can be varified by calculating the transfer function as

$$G(s) = \begin{bmatrix} 0 & \frac{-1}{s-1} \\ \frac{2(s-2)}{(s-3)(s-1)} & \frac{1}{s-3} \\ 0 & \frac{2}{s-3} \end{bmatrix}$$

and its Smith-McMillan form, $M(s)$, as

$$M(s) = \frac{1}{(s-3)(s-1)} \begin{bmatrix} 1 & 0 \\ 0 & s-2 \\ 0 & 0 \end{bmatrix}$$

which, by Definition 1, displays a transmission zero at $z = 2$. □

4.4. Systems Zeros

So far transmission zeros, decoupling zeros, and invariant zeros have been defined for a system. Since in general the invariant zeros don't include all of the decoupling zeros, it is desirable to provide a definition which will include all previously defined zeros of a system.

With this goal in mind, consider a system which gives rise to a system matrix $P(s)$ and a transfer function matrix $G(s)$. The minors of $G(s)$ are related to the minors of $P(s)$ by the following equation [13].

$$G\left({i_1,\ldots,i_p \atop j_1,\ldots,j_p};s\right) = P\left({1,\ldots,n,n+i_1,\ldots,n+i_p \atop 1,\ldots,n,n+j_1,\ldots,n+j_p};s\right)/\det[sI-A] \qquad (4.4,1)$$

To see this relationship, first consider $P(s)$. It can be factored as:

$$P(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} sI-A & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} I_n & 0 \\ C & I_r \end{bmatrix} \begin{bmatrix} I_n & -(sI-A)^{-1}B \\ 0 & C(sI-A)^{-1}B+D \end{bmatrix} \qquad (4.4,2)$$

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Taking determinants of each side yields:

$$\det[P(s)] = \det[sI-A]\det[C(sI-A)^{-1}B + D] \tag{4.4,3}$$

Let c_i denote a row of C, b_j denote a column of B, and d_{ij} an element of D. It follows from (4.4,3) that

$$g_{ij}(s) = c_i(sI-A)^{-1}b_j + d_{ij} = P\left({1,\ldots,n,n+i \atop 1,\ldots,n,n+j}\right)s)/\det[sI-A] \tag{4.4,4}$$

The element $g_{ij}(s)$ can be thought of as a 1×1 minor of G(s). In exactly the same way, this argument can be extended to any minor of G(s) of any order, resulting in (4.4,1).

Equation (4.4,1) suggests the following definition:

<u>Definition 10</u> [24]

The system zeros of a system matrix P(s) are the complex roots of the monic greatest common divisor of all non-zero minors of P(s) of the form

$$P\left({1,2,\ldots,n,n+i_1,n+i_2,\ldots,n+i_p \atop 1,2,\ldots,n,n+j_1,n+j_2,\ldots,n+j_p};s\right)$$

where $n + p$ is the maximum order of a minor of this form which is non-zero. □

Definition 10 and Theorem 5 are similar in that both involve minors of maximum order of P(s). However, Theorem 5 requires that <u>all</u> minors of maximum order be calculated, where as Definition 10 is restricted to the subset of those minors that contain the first n rows and n columns of P(s). When the number of inputs and the number of outputs is the same and the system is non-degenerate, then the invariant zeros are the same as the system zeros because there is only one minor of maximum order. However, when one of these conditions fails to hold,

it would suggest that Definition 10 defines a larger set of zeros than the invariant zeros. The exact relationship is given by the following theorem:

Theorem 9 [24]

a) The set of system zeros of a system matrix $P(s)$ consists of:

i) the set of transmission zeros

ii) the set of input decoupling zeros

iii) the set of output decoupling zeros minus the set of input-output decoupling zeros.

b) The set of invariant zeros is a subset of the system zeros.

Proof: This proof is in the same spirit as the proof of Theorem 8. Consider a system matrix $P(s)$ with normal rank $n + k$. The following result will be needed below: If $n + p$ is the maximal order of a minor of the form in Definition 10, then $p = k$. Note that this also proves part (b) of the theorem, by Theorem 5. The normal rank is the order of the largest non-zero minor. To find one such minor, examine each column in the order $1,2,\ldots,n+m$ and select each column which is linearly independent from those already selected. Since $sI-A$ is non-singular, the first n columns qualify. Exactly k of the remaining columns qualify. Repeat this procedure for the rows of $P(s)$. Again the first n qualify with exactly k remaining rows. This minor has the form required by Definition 10. Hence, $p = k$.

Return to the system matrix $P(s)$. It can be written as a product of matrices as in (4.3,8):

$$P(s) = \begin{bmatrix} Q_1(s) & 0 \\ 0 & I_r \end{bmatrix} \bar{P}(s) \begin{bmatrix} Q_2(s) & 0 \\ 0 & I_m \end{bmatrix}$$

$$= Q_1^*(s)\bar{P}(s)Q_2^*(s) \qquad (4.4,5)$$

Recall that $Q_1(s)$ contains the input decoupling zeros, $Q_2(s)$ contains the output decoupling zeros minus the input-output decoupling zeros, and $\bar{P}(s)$ represents a reachable and observable system. To calculate the system zeros, the formula given in (4.3,9) is used, where the minors are required to be of the type in Definition 10.

$$P\binom{1,\dots,n,n+i_1,\dots,n+i_p}{1,\dots,n,n+j_1,\dots,n+j_p};s) = \sum_{\substack{1\le\alpha_1<\dots<\alpha_{n+p}\le n+r \\ 1\le\gamma_1<\dots<\gamma_{n+p}\le n+m}} Q_1^*\binom{1,\dots,n,n+i_1,\dots,n+p}{\alpha_1,\dots,\alpha_{n+p}};s) .$$

$$\bar{P}\binom{\alpha_1,\dots,\alpha_{n+p}}{\gamma_1,\dots,\gamma_{n+p}};s)Q_2^*\binom{\gamma_1\dots\gamma_{n+p}}{1,\dots,n,n+j_1..,n+j_p};s) \qquad (4.4,6)$$

where $n + p$ is the normal rank of $P(s)$. In this sum (4.4,6), consider a term where there exists an α_s such that $\alpha_s > n$ and $\alpha_s \neq n + i_\ell$ for all ℓ, $1 \le \ell \le p$. Then that minor of $Q_1^*(s)$ is zero, as can be seen from the special form of $Q_1^*(s)$ in (4.4,5). The same argument holds for $Q_2^*(s)$. This implies that the sum in (4.4,6) can be replaced by

$$P\binom{1,\dots,n,n+i_1,\dots,n+i_p}{1,\dots,n,n+j_1,\dots,n+j_p};s) = \det[Q_1(s)]\bar{P}\binom{1,\dots,n,n+i_1,\dots,n+i_p}{1,\dots,n,n+j_1,\dots,n+j_p};s)\det[Q_2(s)] \qquad (4,4,7)$$

Since $\bar{P}(s)$ represents a reachable and observable system, by Theorem 6 the invariant factors of $\bar{P}(s)$ are just the invariant factors of its transfer function. Therefore, $\bar{P}(s)$ can have at most $p \le \min(m,r)$ invariant factors that are not identically 1. It follows that the Smith form $\bar{P}_s(s)$

of $\overline{P}(s)$ is

$$\overline{P}_s(s) = \begin{bmatrix} I_n & 0 \\ 0 & \text{diag}[\epsilon_1(s),\ldots,\epsilon_p(s),0,\ldots,0] \end{bmatrix} \qquad (4.4,8)$$

Using the determental divisor properties of the Smith form (Section 2.3) and (4.4,7) and (4.4,8), it is now seen that the greatest common divisor of the minors in (4.4,7) is

$$\det[Q_1(s)][\epsilon_1(s)\ldots\epsilon_p(s)]\det[Q_2(s)] \qquad (4.4,9)$$

When this polynomial is adjusted to be monic, its roots form a complete set of decoupling zeros (by construction of $Q_1(s)$ and $Q_2(s)$) and all the transmission zeros of $P(s)$. This completes the proof. □

Example 14

Consider the system matrix $P(s)$ introduced in Example 9. In Example 11, all minors of maximum order were calculated to find the invariant zeros. Using Definition 10, the minors of interest are

$$P\left({1,\ldots,8 \atop 1,\ldots,8}, s\right) = -\frac{1}{2}(s+1)(s+4)(s-3)(s-2)$$

$$P\left({1,\ldots,8 \atop 1,\ldots,6,8,9}, s\right) = 4(s-1)(s+1)(s+4)(s-2)$$

The monic greatest common divisor of these minors is

$$(s+1)(s+4)(s-2)$$

so that the system zeros are

$$z = -1,\ z = -4,\ z = 2. \qquad \square$$

Definition 10 can be used to interpret system zeros in another way. Consider

$$P\left(\begin{matrix}1,2,\ldots,n,i_1,\ldots,i_p\\1,2,\ldots,n,j_1,\ldots,j_p\end{matrix};s\right) \qquad (4.4,10)$$

which is a minor of P(s) that satisfies Definition 10. This minor describes a state space representation

$$\begin{aligned}\dot{x} &= Ax + B_p u\\ y &= C_p x + D_p u\end{aligned} \qquad (4.4,11)$$

where the matrices B_p, C_p, and D_p are obtained from the rows and columns indicated in (4.4,10); the input and output spaces now both having dimension p. This system is non-degenerate as (4.4,10) is required to be non-zero. Hence, by Theorem 5, the invariant zeros of (4.4,11) are the roots of (4.4,10). In this light, Definition 10 says that the system zeros of P(s) are those invariant zeros which are common to all non-degenerate subsystems formed by selecting p rows of the C matrix and p columns of the B matrix; i.e. subsystems of the form (4.4,11).

CHAPTER 5

PROPERTIES OF ZEROS

5.1. Introduction

Chapter 4 introduced the definitions of several types of zeros for state space systems and described their interrelationship. Various types of zeros can be calculated from the corresponding definitions or theorems of the last chapter. However, these definitions and theorems, which are based primarily on the Smith form, are not readily adapted to a digital computer and hand calculation is tedious for higher order systems. Therefore, an important task is to present several algorithms for calculating zeros, based on additional properties of zeros, that are easier to apply than using the basic properties, or either, that can be readily adapted to a digital computer.

This chapter is devoted to the development of those properties which will be useful in constructing such algorithms. It should be noted that these properties are important in their own right. In fact, many of these properties are just generalizations of properties of zeros of scalar transfer functions. In particular, it is shown below that invariant zeros are unchanged by state feedback, that they are the limiting positions of the systems poles under high gain feedback, and that they are the poles of an inverse system.

This chapter also examines the geometrical properties of zeros. Although these properties are a consequence of definitions given earlier, the invariant zeros (and transmission zeros) can be defined and developed from a geometrical analysis of the state space representation (2.2,1) or

(2.2,5) of a system. It turns out that the invariant zeros are intimately connected to two well known subspaces, the largest (A,B)-invariant subspace in the null space of C and the largest controllability subspace in the null space of C [12]. The definitions of these subspaces are generalized to systems where $D \neq 0$ and the connection with invariant zeros is made explicit. These geometrical ideas also form the basis of a new algorithm (presented in Chapter 7 below) for calculating these two subspaces, and, hence, the invariant zeros of the system.

The geometrical analysis given here applies equally well to continuous systems (2.2,1) or discrete systems (2.2,5). Therefore, to introduce generality in the notation, either of these two types of systems will be denoted (A,B,C,D). Some of the discussion of the geometrical properties is done in terms of discrete systems. This is for pedagogical reasons (as will be clear below). The analysis, however, applies just as well to continuous systems.

Section 2 discusses in detail the geometrical ideas needed for the discussion of invariant zeros. Two important subspaces, the unknown-input unobservable subspace and the null-output reachable subspace, are introduced and their relevant properties are presented. It is shown that a system (A,B,C,D) can be placed in a canonical form to exhibit the properties of these subspaces. This analysis will be used again in Chapter 7 below.

The specific relationship of invariant zeros to the geometric properties of the system is given in Section 3. This is accomplished via the canonic form introduced in Section 2. Several other geometrical properties of invariant zeros is also discussed.

Section 4 describes various properties of system zeros and invariant zeros which will be useful in discussing algorithms for calculating zeros. These properties include the invariance of system zeros under the coordinate changes of the input space, state space, and output space and under output feedback. The invariant zeros are shown to be unchanged by state feedback and to be the limiting positions of the systems poles under high gain feedback.

Section 5 reviews the notion of an inverse system and develops the basic theory behind this idea. Such a system is then constructed and it is shown that the poles of an inverse system are the invariant zeros of the original system. The geometrical ideas of Section 5.2 play a key role here.

5.2. Some Geometrical Aspects of Linear Systems

The definition and properties of two subspaces relevant to the discussion of zeros are presented in this section. These subspaces were first introduced in [25] and the notation and definitions here follow that references.

The first subspace of interest is the unknown-input unobservable subspace, denoted by $\mathcal{L}$ throughout this paper.

Definition 11 [25]

The vector ξ is an element of $\mathcal{L}_i$, the ith unknown-input unobservable subspace, if there exists a control sequence

$$(u_0, u_1, \ldots, u_{i-1})$$

such that if $x_o = \xi$ then

$$y_o = y_1 = \dots = y_{i-1} = 0$$

By definition, $\mathcal{L}_o = \mathcal{X}$, the whole state space. □

Unknown-input unobservable subspaces have several interesting properties [25]. Two basic properties are given by the following lemma:

<u>Lemma 1</u> [25]

a) $\mathcal{L}_i \supset \mathcal{L}_{i+1}$ for all i

b) For some $k \leq n$, $\mathcal{L}_k = \mathcal{L}_{k+j}$; for $j = 1,2,\dots$

Proof: Part (a) If $\xi \in \mathcal{L}_{i+1}$, then there exists a control sequence

$$(u_0, u_1, u_2, \dots, u_i) \tag{5.2,1}$$

such that

$$y_0 = y_1 = \dots = y_i = 0 \tag{5.2,2}$$

But then, $\xi \in \mathcal{L}_i$ for the same control, sequence (5.2,1) by simply dropping u_i from the sequence.

Part (b). Let

$$\tau_i = d(\mathcal{L}_i) \tag{5.2,3}$$

Then

$$\tau_0 = n \text{ and } \tau_i \geq \tau_{i+1} \tag{5.2,4}$$

as can be seen from Part (a). Let k be the first i such that

$$\tau_k = \tau_{k+1} \tag{5.2,5}$$

Again, by Part (a) and (5.2,5)

$$\mathcal{L}_k = \mathcal{L}_{k+1} \tag{5.2,6}$$

Clearly $k \leq n$. It must be shown that

$$\mathcal{L}_k = \mathcal{L}_{k+j} \tag{5.2,7}$$

for all $j = 1,2,\ldots.$ The argument is by induction. The hypothesis (5.2,7) for $j = 1$ is shown in (5.2,6). Suppose that (5.2,7) holds for some ℓ_j, i.e. $\xi \in L_{k+\ell}$ and (5.2,7) holds for $y = 1,\ldots,\ell$ for some control sequence

$$(u_0, u_1, \ldots, u_{k+(\ell-1)}) \tag{5.2,8}$$

This control sequence produces a state trajectory

$$(x_0, x_1, \ldots, x_{k+(\ell-1)}, x_{k+\ell}) \tag{5.2,9}$$

Let

$$\alpha = x_{k+(\ell-1)} \in \mathcal{L}_{k+(\ell-1)} \tag{5.2,10}$$

By induction hypothesis

$$\mathcal{L}_{k+(\ell-1)} = \mathcal{L}_{k+\ell} \tag{5.2,11}$$

so that $\alpha \in \mathcal{L}_{k+\ell}$. This implies there exist a control sequence

$$(u_0', u_1', \ldots, u_{k+(\ell-1)}') \tag{5.2,12}$$

such that $x_0 = \xi$ and $\alpha = x_{k+\ell}$. But then the application of the control sequence

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$$(u_0', u_1', \ldots, u_{k+(\ell-1)}', u_{k+(\ell-1)}) \qquad (5.2,13)$$

where the last term in (5.2,13) is the last turn in (5.2,8), shows that $\xi \in \mathcal{L}_{k+(\ell+1)}$. Therefore, (5.2,7) follows. □

This shows that after no more than n steps, the unknown-input unobservable subspaces are all the same. In the sequel, the term <u>unknown-input unobservable subspace</u> shall refer to this subspace.

The subspaces $\mathcal{L}_i$, $i = 0,1,2,\ldots$ can be characterized in terms of the matrices (A,B,C,D). For this purpose, consider the following algorithm:

<u>Algorithm 1</u> [26]

Step 1: For $i = 0$, set $M_0 = 0$ where M_0 is a $1 \times n$ matrix.

Step 2: Form the matrix

$$\Gamma_i = \begin{bmatrix} M_i B & M_i A \\ D & C \end{bmatrix} \qquad (5.2,14)$$

and carry out the indicated multiplication. Then find a non-singular matrix S_i such that

$$S_i \Gamma_i = \begin{array}{c} \begin{bmatrix} F_{i+1} & G_{i+1} \\ 0 & M_{i+1} \end{bmatrix} \\ \begin{array}{cc} m \quad & \quad n \end{array} \end{array} \qquad (5.2,15)$$

where F_{i+1} has full row rank.

Step 3: Set $i = i+1$ and go to step 2. □

The following theorem relates the subspaces $\mathcal{L}_i$ to Algorithm 1:

Theorem 10 [26]

Let M_i be the matrices obtained from Algorithm 1. Then

$$\mathcal{L}_i = \mathcal{N}[M_i]; \qquad i = 0,1,2,\ldots$$

Proof: Consider any $\xi = x_o$ and any control sequence

$$(u_0, u_1, \ldots, u_{i-1}) \tag{5.2,16}$$

applied to a system (2.2,5). It follows directly from the repeated application of (2.2,5) that

$$\begin{bmatrix} M_0x_i \\ y_{i-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} M_0A^i & M_0A^{i-1}B & \cdots & M_0AB & M_0B \\ CA^{i-1} & CA^{i-2}B & \cdots & CB & D \\ \vdots & \vdots & & D & 0 \\ CA & CB & \iddots & \vdots & \vdots \\ C & D & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ u_o \\ \vdots \\ u_{i-1} \end{bmatrix} \tag{5.2,17}$$

for any M_0. Notice that the blocks in the first two rows of (5.2,17) can be associated with Γ_0 (5.2.14) in Algorithm I. Multiply the first two rows by S_0((5.2,14)) and use (5.2,15). Now (5.2,17) has the form

$$\begin{bmatrix} S_0\begin{bmatrix} M_0x_i \\ y_{-1} \end{bmatrix} \\ y_{i-2} \\ \vdots \\ y_0 \end{bmatrix} = \begin{bmatrix} G_1A^{i-1} & G_1A^{i-2}B & \cdots & G_1B & F_1 \\ M_1A^{i-2} & M_1A^{i-3}B & & M_1B & 0 \\ CA^{i-2} & CA^{i-3} & \cdots & D & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ C & D & & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ u_0 \\ \vdots \\ u_{i-1} \end{bmatrix} \tag{5.2,18}$$

It is seen that the blocks in the second and third row are associated with Γ_1 (5.2,14). Again, multiply by S_1 (5.2,15) and use (5.2,15). Continue this process for i steps. By the non-singularity of the matrices S_i and the fact that $M_0 = 0$, it follows that

$$y_0 = y_1 = \dots = y_{i-1} = 0 \tag{5.2,19}$$

if and only if

$$\begin{bmatrix} G_1A^{i-1} & G_1A^{i-2} & \dots & G_1B & F_1 \\ G_2A^{i-2} & \dots & & F_2 & 0 \\ \vdots & \vdots & & & \vdots \\ G_i & F_i & & & \vdots \\ M_i & 0 & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} \xi \\ u_0 \\ \vdots \\ \vdots \\ u_{i-1} \end{bmatrix} = 0 \tag{5.2,20}$$

If ξ is an element of $\mathcal{L}_i$, then by Definition 11, (5.2,19) holds for some control sequence (5.2,16). For that control sequence it follows from (5.2,20) that

$$M_i\xi = 0 \tag{5.2,21}$$

Conversely, assume that (5.2,21) holds. Then since the matrices F_j have full row rank, there exists some control sequence (5.2,16) such that (5.2,20) is satisfied. This implies that (5.2,19) holds and by Definition 11, $\xi \in \mathcal{L}_i$. Thus the theorem is proved. □

Algorithm 1 immediately yields the first property of the subspaces $\mathcal{L}_i$ which is fundamental to what follows:

<u>Lemma 2</u> [26]

The subspace $\mathcal{L}_i$ is unaffected by state feedback.

Proof: Consider a system defined by the matrices (A+BK,B,C+DK,D), from (5.2,15) it is seen that

$$S_i\Gamma_i = S_i\begin{bmatrix} M_iB & M_i(A+BK) \\ D & C+DK \end{bmatrix} = S_i\begin{bmatrix} M_iB & M_iA \\ D & C \end{bmatrix}\begin{bmatrix} I_m & K \\ 0 & I_n \end{bmatrix}$$

$$= \begin{bmatrix} F_{i+1} & G_{i+1} \\ 0 & M_{i+1} \end{bmatrix}\begin{bmatrix} I_m & K \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} F_{i+1} & G_{i+1}+F_{i+1}K \\ 0 & M_{i+1} \end{bmatrix} \tag{5.2,22}$$

Thus, by Theorem 10 the subspace $\mathcal{L}_i$ are the same for the system (A,B,C,D), with or without feedback. □

That the subspaces $\mathcal{L}_i$ are not changed by state feedback suggests that they might be very powerful for describing the structure of a system. Indeed, its first important consequence is embodied in the following theorem.

<u>Theorem 11</u> [26]

Consider state feedback of the form

$$u_i = K^* x_i + v_i \tag{5.2,23}$$

where

$$K^* = -F^+_{n+1}G_{n+1}$$

and F^+_{n+1} is the pseudo-inverse of F_{n+1}. Then the system

$$\begin{bmatrix} x_{i+1} \\ y_i \end{bmatrix} = \begin{bmatrix} A+BK^* & B \\ C+DK^* & D \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

after the appropriate similarity transformations, takes on the canonical form:

$$\begin{bmatrix} \bar{x}_{i+1} \\ \tilde{x}_{i+1} \\ \hline y_i \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ A_3 & A_4 & B_3 & B_4 \\ \hline C_1 & 0 & D_1 & 0 \end{array}\right] \begin{bmatrix} \bar{x}_i \\ \tilde{x}_i \\ \bar{v}_i \\ \tilde{v}_i \end{bmatrix} \tag{5.2,24}$$

where

$$\mathcal{L}_n = \left\{ \begin{bmatrix} 0 \\ \\ \tilde{x}_i \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{N}[F_{i+1}] = \left\{ \begin{bmatrix} 0 \\ \\ \bar{v}_i \end{bmatrix} \right\}$$

Proof: From (5.2,15)

$$S_n \begin{bmatrix} M_n(Ax+Bu) \\ \\ Cx + Du \end{bmatrix} = \begin{bmatrix} F_{n+1}u + G_{n+1}x \\ \\ M_{n+1}x \end{bmatrix} \tag{5.2,25}$$

for all (x,u). Note that

$$\mathcal{N}[M_n] = \mathcal{N}[M_{n+1}] \tag{5.2,26}$$

by Lemma 1 and Theorem 10. Consider the feedback defined in (5.2,23). Substituting (5.2,23) into (5.2,25), it is seen that for all $v \in \mathcal{N}[F_{n+1}]$

and $x \in \mathcal{L}_n$

$$\begin{bmatrix} M_n(Ax+Bu) \\ \\ Cx+Du \end{bmatrix} = 0 \qquad (5.2,27)$$

(By Lemma 2, this substitution doesn't change $\mathcal{L}_n$). It follows that $(Ax+Bu)$ is an element of $\mathcal{L}_n$. This leads to the representation (5.2,24) in the appropriate bases. □

The canonic form (5.2,24) shows that $\mathcal{L}_n$ is (A,B)-invariant (See Section 2.1). In fact, $\mathcal{L}_n$ is a generalization of Wonham's maximal (A,B)-invariant subspace in the null space of C [12]. The relationship is made precise by the following definition and theorem.

Definition 12 [26]

A subspace $\mathcal{V}$ is a null-output (A,B)-invariant subspace if for every $x \in \mathcal{V}$ there exists some u such that $(Ax+Bu) \in \mathcal{V}$ and $(Cx+Du) = 0$.

Theorem 12 [26]

A maximal null-output (A,B)-invariant subspace, denoted $\mathcal{L}^*$, exists, and $\mathcal{L}_n = \mathcal{L}^*$ where $\mathcal{L}^*$ is $(A+BK^*)$-invariant.

Proof: From Definitions 11 and 12, it is seen that $\mathcal{U} \subset \mathcal{L}_n$ for every $\mathcal{U}$, $\mathcal{U}$ a null-output (A,B)-invariant subspace. Conversely, take $u = K^*x$ and consider $x \in \mathcal{L}_n$. From canonic form (5.2,24), it is obvious that $\mathcal{L}_n$ satisfies Definition 12. □

The subspace $\mathcal{L}^*$ has been very useful in geometric control theory [12]. It will play a fundamental role in what follows.

The second subspace of interest is the null-output reachable space.

Definition 13 [25]

The vector ξ is an element of $\mathcal{R}_i$, the ith null-output reachable space, if there exists a control sequence

$$(u_0, u_1, \ldots, u_{i-1})$$

such that if $x_0 = 0$ then $x_i = \xi$ and

$$y_0 = y_1 = \ldots y_{i-1} = 0 \qquad \square$$

Like $\mathcal{L}_i$, $\mathcal{R}_i$ has nesting properties and a limit condition which are given in the next lemma. The proof is similar to Lemma 1 and is given in [22].

Lemma 3 [25]

a) $\mathcal{R}_i \subset \mathcal{R}_{i+1} \qquad i = 0,1,2,\ldots$

b) For some $k \leq n$, $\mathcal{R}_k = \mathcal{R}_{k+j}$ for $j = 1,2,\ldots$ $\square$

The null-output reachable subspaces are related to another fundamental subspace, the maximal reachability subspace in the null space of C. To establish this connection, define the reachable subspace $\mathcal{R}$ of the pair (A,B) as

$$\mathcal{R} = \langle A | \mathcal{R}[B] \rangle = \mathcal{R}[B] + A\,\mathcal{R}[B] + \ldots + A^{n-1}\mathcal{R}[B] \qquad (5.2,28)$$

if $d(\mathcal{X}) = n$. If $\mathcal{R} = \mathcal{X}$, it is said that (A,B) is reachable. Certain subspaces of the reachable space turn out to be of interest in geometric control theory. In general, a subspace $\mathcal{R}$ is a reachability subspace if there exist maps $K: \mathcal{X} \to \mathcal{U}$ and $J: \mathcal{U} \to \mathcal{U}$ such that

$$\mathcal{R} = \langle (A+BK) \mid \mathcal{R}[BJ] \rangle \tag{5.2,29}$$

The reachable subspace is a reachability subspace as can be seen by taking $K = 0$ and $J = I_m$. Note that reachability subspaces are (A,B)-invariant subspaces by (5.2,28) and the definition of (A,B)-invariant subspaces (Section 2.1). The particular reachability subspace of interest here is defined next.

Definition 14 [26]

The subspace $\mathcal{R}$ is a null-output reachability subspace if there exist matrices K and J such that

$$\mathcal{R} = \langle (A+BK) \mid \mathcal{R}[BJ] \rangle$$

$$(C+DK)\mathcal{R} = 0$$

$$D\mathcal{R}[J] = 0 \qquad \square$$

The relationship between $\mathcal{L}_n$, $\mathcal{R}_n$, and the null-output reachability space is given by the following theorem.

Theorem 13 [26]

A maximal null-output reachability space, denoted $\mathcal{R}^*$, exists and

$$\mathcal{R}^* = \mathcal{R}_n \cap \mathcal{L}_n$$

It is generated by K^* (5.2,23) and J where

$$\mathcal{R}[J] = \mathcal{N}[F_{n+1}]$$

Proof: Consider any null-output reachability subspace $\mathcal{R}$. It follows from Definitions 13 and 14 and Lemma 2 that $\mathcal{R} \subset \mathcal{R}_n$. Also, by reachability,

$(A+BK)\mathcal{R} \subset \mathcal{R}$, and $(C+DK)\mathcal{R} = 0$ imply that $\mathcal{R} \subset \mathcal{L}^*$ by Definition 12 and Theorem 12. It follows that

$$\mathcal{R} \subset \mathcal{R}_n \cap \mathcal{L}_n \qquad (5.2,30)$$

If it can be shown that $\mathcal{R}_n \cap \mathcal{L}_n$ also satisfies Definition 14, then (5.2,30) shows the $\mathcal{R}_n \cap \mathcal{L}_n$ is the required maximal null-output reachability space by part (b) of Lemma's 1 and 2. First, $\mathcal{R}_n \cap \mathcal{L}_n$ is characterized in terms of canonic form (5.2,24):

$$\mathcal{R}_n \cap \mathcal{L}_n = \left\{ \begin{bmatrix} 0 \\ \tilde{x}_i \end{bmatrix} : \ \tilde{x}_i \in \langle A_4 | B_4 \rangle \right\} \qquad (5.2,31)$$

Note that the right side is a subset of the left, as can be seen directly from (5.2,24). Now consider any $x_{i+1} \in \mathcal{R}_n \cap \mathcal{L}_n$. From (5.2,24) it follows that

$$\bar{x}_{i+1} = 0 \text{ and } y_i = 0 \qquad (5.2,32)$$

With reference to (5.2,24), it is seen that

$$\eta \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = 0 \qquad (5.2,33)$$

This in turn implies that

$$\bar{x}_i = 0 \text{ and } \bar{v}_i = 0 \qquad (5.2,34)$$

This argument can be used to show that

$$\bar{x}_j = 0, \; \bar{v}_j = 0 \quad j = i, i-1, \ldots, 0 \tag{5.2,35}$$

Hence, it has been shown that

$$\mathcal{R}_n \cap \mathcal{L}_n \subset \left\{ \begin{bmatrix} 0 \\ \tilde{x}_i \end{bmatrix} : \tilde{x}_i \in \langle A_4 | B_4 \rangle \right\} \tag{5.2,36}$$

which establishes (5.2,31). Finally, from the construction of canonic form (5.2,24) it follows that (5.2,31) satisfies Definition 14 for

$$K = K^* \quad \text{and} \quad \mathcal{R}[J] = \mathcal{N}[F_{n+1}] \tag{5.2,37}$$

□

Since $\mathcal{R}^*$ is a (A,B)-invariant subspace for K^* (Definition 12), it follows that canonic form (5.2,24) would admit an even more explicit representation in the appropriate state and control bases.

Corollary 1 [27]

Canonic form (5.2,24), in the appropriate state and control bases takes on the form

$$\left[\begin{array}{c} x_{i+1} \\ \hat{x}_{i+1} \\ \tilde{x}_{i+1} \\ \hline y_i \end{array}\right] = \left[\begin{array}{ccc|cc} A_1 & 0 & 0 & B_1 & 0 \\ \hat{A}_3 & A_5 & A_6 & \hat{B}_3 & B_5 \\ \tilde{A}_3 & 0 & A_7 & \tilde{B}_3 & 0 \\ \hline C_1 & 0 & 0 & D_1 & 0 \end{array}\right] \begin{bmatrix} \bar{x}_i \\ \hat{x}_i \\ \tilde{x}_i \\ \bar{v}_i \\ \hat{v}_i \end{bmatrix} \tag{5.2,38}$$

where (A_5, B_5) is reachable. □

Interperted geometrically, canonic form (5.2,38) exhibits a direct sum decomposition of state space

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \hat{\mathcal{X}} \oplus \tilde{\mathcal{X}} \tag{5.2,39}$$

where

$$\begin{aligned} &\hat{\mathcal{X}} = \mathcal{R}^* \\ &\hat{\mathcal{X}} \oplus \tilde{\mathcal{X}} = \mathcal{L}^* \\ &\bar{\mathcal{X}} \text{ is isomorphic to } \mathcal{X}/\mathcal{L}^* \\ &\tilde{\mathcal{X}} \text{ is isomorphic to } \mathcal{L}^*/\mathcal{R}^* \end{aligned} \tag{5.2,40}$$

The invariant zeros can now be described in terms of canonic form (5.2,38) and the state space decomposition (5.2,40). This is undertaken in the next section.

5.3. Geometric Properties of Invariant Zeros

Recall that the invariant zeros are the roots of the invariant polynomials of the system matrix P(s) (Definition 5). The invariant zeros are closely related to canonic form (5.2,38) developed in Section 5.2. This connection is formally established by the methods of Chapter 4 applied to the results of the geometric analysis in Section 5.2.

The following lemma will be needed in the proof of the main theorem below.

Lemma 4 [28]

Consider a system with system matrix P(s). If

$$\mathcal{N}[\Gamma_{n+1}] = 0$$

where Γ_n is defined in Algorithm 1 (Section 5.2), then there exists a polynomial matrix W(s) such that

$$W(s)P(s) = I$$

This implies that

$$P(s) \approx \begin{bmatrix} I \\ \\ 0 \end{bmatrix}$$

where "$\approx$" in this work means "has the same Smith form as". □

Proof: See Reference [28]. From (5.2,15) it is seen that

$$\mathcal{N}[\Gamma_{n+1}] = 0 \text{ implies that}$$

$$\mathcal{N}[M_{n+1}] = 0 = \mathcal{L}_{n+1} = \mathcal{L}^* \qquad (5.3,1)$$

from Theorem 10 and 12, and

$$\mathcal{N}[F_{n+1}] = 0 \qquad (5.3,2)$$

By Lemma 4, such a system has no invariant zeros. The idea of the following analysis is to identify a subsystem of a given system with these properties.

Given a system, bring it into canonic form (5.2,38). As was discussed in Section 5.2, this is done by applying feedback of the form

$$u = K^* x + v \qquad (5.3,3)$$

and transforming the state and input spaces (Corollary 1). Let these transformations be given by

$$z = Tx \text{ and } u = Gv \qquad (5.3,4)$$

where T and G are non-singular matrices of the appropriate size. The transformed system has a system matrix $\bar{P}(s)$ that is given by

$$\bar{P}(s) = \left[\begin{array}{ccc|cc} sI-A_1 & 0 & 0 & -B_1 & 0 \\ -\hat{A}_3 & sI-A_5 & -A_6 & -\hat{B}_3 & -B_5 \\ -\tilde{A}_3 & 0 & sI-A_7 & -\tilde{B}_3 & 0 \\ \hline C_1 & 0 & 0 & D_1 & 0 \end{array}\right] \qquad (5.3,5)$$

The transformed system is obtained by substituting (5.3,3) and (5.3,4) into the original state equations. This gives the following relationship between P(s) and $\bar{P}(s)$:

$$\begin{bmatrix} T & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & 0 \\ K^* & I_m \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & G \end{bmatrix} = \bar{P}(s) \qquad (5.3,6)$$

This can be checked by straight forward calculation. The matrices pre-and post-multiplying P(s) can be interpreted as elementary row and column operations. It follows from the comments is Section 2.3 on the Smith form that

$$P(s) \approx \bar{P}(s) \qquad (5.3,7)$$

By using elementary row and column operations, (5.3,5) can be written as

$$\bar{P}(s) \approx \left[\begin{array}{cc|cc|c} sI-A_1 & -B_1 & 0 & 0 & 0 \\ C_1 & D_1 & 0 & 0 & 0 \\ \hline -\hat{A}_3 & -\hat{B}_3 & sI-A_5 & -B_5 & -A_6 \\ \hline -\tilde{A} & -\tilde{B}_3 & 0 & 0 & sI-A_7 \end{array}\right] \tag{5.3,8}$$

When Algorithm 1 is applied to the subsystem (A_1,B_1,C_1,D_1), it is found that

$$\mathcal{N}[\Gamma_{n+1}] = 0 \tag{5.3,9}$$

This follows from the construction of canonic form (5.2,24). (See proof of *Theorem* 11). *It also follows* from the discussion of decoupling zeros in Section 4.3 and the fact that (A_5,B_5) is reachable, that

$$[sI-A_5 \ \vdots \ -B_5] \approx [I \quad 0] \tag{5.3,10}$$

Hence, by Lemma 4 and equation (5.3,10), it follows that

$$\bar{P}(s) \approx \left[\begin{array}{c|cc|c} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & sI-A_7 \end{array}\right] \approx P(s) \tag{5.3,11}$$

Now, it is clear from Definition 5 that the following theorem has been proved.

<u>Theorem 14</u> [28]

The invariant zeros are the roots of the invariant polynomials of $(sI-A_7)$ in canonic form (5.3,8). □

The proof of Theorem 14 relied on the system matrix characterization of invariant zeros given in Definition 5. However, a description of invariant zeros can be formulated from a purely geometrical point of view as follows [29]. Let K be a feedback matrix such that

$$(A+BK)\mathcal{L}^* \subset \mathcal{L}^* \tag{5.3,12}$$

Denote by A_k the map (A+BK) induced in $\mathcal{X}/\mathcal{R}^*$. The invariant zeros are the roots of the invariant factors of the map A_k restricted to $\mathcal{L}^*/\mathcal{R}^*$. (Compare to the remark at the end of Section 5.2).

There is also a geometric connection with the system matrix. Consider a system matrix P(s) with full normal rank. By Theorem 4, z is an invariant zero of P(s) if

$$\rho[P(z)] < \rho_n[P(\cdot)] \tag{5.3,13}$$

This implies that

$$\eta[P(z)] \neq 0 \tag{5.3,14}$$

Therefore, there exist non-zero vectors x_o and g, elements of the state space and input space respectively such that

$$P(z)\begin{bmatrix} x_o \\ \\ g \end{bmatrix} = 0 \tag{5.3,15}$$

The vectors x_o and g are called the <u>invariant zero direction vectors</u> for z,

or zero directions, for short [17]. Suppose that the system represented by $P(s)$ is excited by an input of the form

$$u(t) = g \exp(zt), \tag{5.3,16}$$

that z is not an input decoupling zero, and z has order 1. If the initial condition of the system is x_o, then the response in state space will be given by

$$x(t) = x_o \exp(zt) \tag{5.3,17}$$

but the output, $y(t)$, will be identically zero [17]. (Compare to the introduction in Section 4.2).

Now suppose that the matrix $P(s)$ has p distinct invariant zeros, $z_1,\ldots,z_p$, of order 1, for simplicity. Then for each z_i there exists a state zero direction x_o^i. These state zero directions x_o^i have a number of interesting properties. For instance, the state zero directions are unaffected by state feedback of the form $u = Kx + v$ [3], and the subspace spanned by the state zero directions is isomonphic to $\mathcal{L}^*/\mathcal{R}^*$ as can be seen from the geometric interpretation of Theorem 14 above. For a more complete discussion of zero directions see [3], [17] and [30].

5.4. Properties of Zeros

The following properties of zeros are important in the construction of algorithms for calculating zeros. The first theorem characterizes the basic properties of system zeros (Definition 10).

Theorem 15 [24]

a) System zeros are unaffected by

1) Input space transformations

2) Output space transformation

3) State space transformations

4) Linear output feedback, $u = Fy + v$

b) The system zeros of a system (A,B,C,D) are the same as those of the dual system (A^T, C^T, B^T, D^T).

Remark: Since transmission zeros (Definition 1), invariant zeros (Definition 5), and decoupling zeros (Definitions 7-9) are all subsets of system zeros (Theorem 9), it follows that this theorem applies to these subsets, too.

Proof of Theorem 15: Consider a system matrix P(s). Let T, G, V be non-singular transformation matrices such that

$$x = Tx'$$

$$u = Gu' \tag{5.4,1}$$

$$y' = Vy$$

where the equations in (5.4,1) define the primed variables. Applying these transformations to the original system, a new system is obtained, represented by the system matrix $\overline{P}(s)$ where

$$\bar{P}(s) = \begin{bmatrix} T^{-1} & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & G \end{bmatrix} \qquad (5.4,2)$$

as can be varified by direct calculation. This shows that $\bar{P}(s)$ can be obtained from P(s) by elementary row and column operations. Hence, $\bar{P}(s)$ has the same Smith form as P(s), and, therefore, the same invariant zeros. If P(s) has no decoupling zeros, the invariant zeros are just the transmission zeros (Theorem 6) which shows the transmission zeros are unchanged by the transformations in (5.4,1). Similar arguments show that

$$\begin{bmatrix} sI-A \\ C \end{bmatrix} \approx \begin{bmatrix} sI-T^{-1}AT \\ VCT \end{bmatrix} \qquad (5.4,3)$$

and

$$[sI-A \mid -B] \approx [sI-T^{-1}AT \mid -TBG] \qquad (5.4,4)$$

So the decoupling zeros are also unchanged. Combining these results with Theorem 9, 1-3 of part (a) of the theorem follow.

Now consider linear output feedback of the form

$$u = Fy + v \qquad (5.4,5)$$

where F is a m x r matrix. The closed loop system can be represented by the system matrix P'(s) where

$$
P'(s)\left[\begin{array}{c} x(s) \\ u(s) \\ y(s) \\ \hline v(s) \end{array}\right] = \left[\begin{array}{ccc|c} sI-A & -B & 0 & 0 \\ C & D & -I_r & 0 \\ 0 & I_m & -F & -I_m \\ \hline C & D & 0 & 0 \end{array}\right]\left[\begin{array}{c} x(s) \\ u(s) \\ y(s) \\ \hline v(s) \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline y(s) \end{array}\right] \qquad (5.4,6)
$$

By using elementary row and column operations, the following reduction of P'(s) is obtained:

$$
P'(s) \approx \begin{bmatrix} sI-A & -B & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & I_m \\ C & D & 0 & 0 \end{bmatrix} \qquad (5.4,7)
$$

and then

$$
P'(s) \approx \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & sI-A & -B \\ 0 & 0 & C & D \end{bmatrix} \qquad (5.4,8)
$$

Therefore, the invariant polynomials of P'(s) are the same as the invariant polynomials of P(s). Hence, the invariant zeros (Definition 5) of the system after the application of feedback (5.4,5) are the same as the invariant zeros of the original system. As remarked above, it follows that the transmission zeros are also unchanged. Similarly, it follows that

$$\left[\begin{array}{ccc|c} sI-A & -B & 0 & 0 \\ C & D & I_r & 0 \\ 0 & I_m & -F & I_m \end{array}\right] \approx \begin{bmatrix} sI-A & -B & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}$$

and (5.4,9)

$$\begin{bmatrix} sI-A & -B & 0 \\ C & D & -I_r \\ 0 & I_m & -F \\ C & D & 0 \end{bmatrix} \approx \begin{bmatrix} sI-A & 0 & 0 \\ 0 & 0 & I_r \\ 0 & I_m & 0 \\ C & 0 & 0 \end{bmatrix} \tag{5.4,10}$$

This shows that the input decoupling zeros (Definition 7) and the output decoupling zeros (Definition 8), respectively, are unchanged. Again, by Theorem 9, (4) of part (a) of the theorem follows.

These arguments can easily be used to show that the system zeros of P(s) are also the system zeros of the dual system $P^T(s)$ where

$$P^T(s) = \begin{bmatrix} sI-A^T & C^T \\ -B^T & D^T \end{bmatrix} \tag{5.4,11}$$

The difference is that the input (output) decoupling zeros of (A,B,C,D) become output (input) decoupling zeros of (A^T,C^T,B^T,D^T). □

Now consider state feedback of the form

$$u = Kx + v \tag{5.4,12}$$

where K is a $m \times n$ matrix. Since the observability properties of a system are not invariant under state feedback, in general the system zeros will change with the application of (5.4,12). However, the invariant zeros

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remain the same.

Theorem 16

The invariant zeros are unaffected by state feedback of the form in (5.4,12).

Proof: This theorem follows directly from Lemma 2 and Theorem 12, which say that $\mathcal{L}^*$ is unaffected by state feedback, and the geometrical interpretation of invariant zeros from Theorem 14. □

For an alternative proof, see [3].

The application of feedback of the form (5.4,12) to a system (A,B,C,D) generates a new system (A+BK,B,C+DK,D). It is possible to call the original system the open loop system and the new system, the closed loop system. Then Theorem 16 says that the open loop invariant zeros and the closed loop invariant zeros are the same, which is analogous to the well known classical result. A similar statement could be made about output feedback and system zeros.

Another property of zeros of a scalar transfer function is that under high gain feedback, the poles of the system tend toward the zeros, or toward infinity as the gain tends toward infinity. There is also a generalization of this result for multivariable systems [31][32]. Consider a system

$$\dot{x} = Ax + Bu \tag{5.4,13a}$$

$$y = Cx + Du \tag{5.4,13b}$$

where the system is non-degenerate; i.e. the system matrix has full rank. Form a closed loop system by using output feedback of the form

$$u = \rho K y \qquad (5.4,14)$$

where ρ is a scalar and K is an arbitrary matrix with rank min(r,m). Substitute (5.4,14) into (5.4,13b) and solve for u

$$u = (\frac{1}{\rho} I_r - KD)^{-1} KCx \qquad (5.4,15)$$

Substitute (5.4,15) into (5.4,13a) to obtain the closed loop system

$$\dot{x} = (A + B(\frac{1}{\rho} I_r - KD)^{-1} KCx \qquad (5.4,16)$$

In [32], the following theorem is proved:

Theorem 17

Given a system (5.4,13), form a closed loop system using (5.4,14). Then if $m = r$, the finite eigenvalues of

$$A + B(\frac{1}{\rho} I_r - KD)^{-1} KC \qquad (5.4,17)$$

as $\rho \to \infty$ coincide with the invariant zeros of (5.4,13). If $m \neq r$, then for "almost all" matrices K, the invariant zeros of (5.4,13) will be contained in the finite eigenvalues of (5.4,17) as $\rho \to \infty$. □

Of course, this theorem holds for discrete systems, too.

5.5. System Inverses

Consider systems of the form

$$x_{i+1} = Ax_i + Bu_i \qquad (5.5,1a)$$

$$y_i = Cx_i + Du_i \qquad (5.5,1b)$$

where $u \in R^m$, $x \in R^n$, $y \in R^r$ and the matrices (A,B,C,D) are defined appropriately. This system can be solved in a straight forward way to give

$$y_i = \left(\sum_{j=0}^{i} CA^{j-1}Bu_{i-j}\right) + CA^i x_0 \tag{5.5,2}$$

Introduce the following notation:

$$u_{[0,i]} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_i \end{bmatrix} \qquad y_{[0,i]} = \begin{bmatrix} y_i \\ y_{i-1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_0 \end{bmatrix} \tag{5.5,3}$$

Form the $(i+1)r \times (i+1)m$ matrix $Q_{[0,i]}$ and the $(i+1)r \times n$ matrix $Q_{[x_0,i]}$

$$Q_{[0,i]} = \begin{bmatrix} CA^{i-1}B & CA^{i-2}B & \dots & CB & D \\ CA^{i-2}B & CA^{i-3}B & \dots & D & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ CB & D & & \cdot & \cdot \\ & & & & \cdot \\ D & 0 & \dots & \dots & 0 \end{bmatrix}, Q_{[x,i]} = \begin{bmatrix} CA^i \\ CA^{i-1} \\ \cdot \\ \cdot \\ \cdot \\ CA \\ C \end{bmatrix} \tag{5.5,4}$$

Finally define the $(i+1)r \times (n+(i+1)m)$ matrix Q_i by

$$Q_i \triangleq [Q_{[x_0,i]} \vdots Q_{[0,i]}] \tag{5.5,5}$$

Now for any input $u_{[0,i]}$ and any initial condition x_0 the output sequence $y_{[0,i]}$ is given by

$$y_{[0,i]} = Q_i \begin{bmatrix} x_0 \\ \\ u_{[0,i]} \end{bmatrix} = Q_{[x_0,i]}x_0 + Q_{[0,i]}u_{[0,i]} \qquad (5.5,6)$$

Note that (5.5,2) is just the first row of blocks in (5.5,6).

The matrix Q_i can be thought to represent a input-output map θ, parameterized by x_0, which sends input sequences $u_{[0,i]}$ to output sequences $y_{[0,i]}$; i.e.

$$y_{[0,i]} = \theta(x_0, u_{[0,i]}) = \theta_1(x_0) + \theta_2(u_{[0,i]}) \qquad (5.5,7)$$

for all i. The operator θ maps infinite sequences of points in R^m to infinite sequences of points in R^r. Define the input space U and the output space Y accordingly.

A left inverse is any operator θ_L: $Y \to U$ such that

$$\theta_L y_{[0,i]} = \theta_L \circ \theta(x_0, u_{[0,i]}) = \theta_L \circ \theta_1(x_0) + \theta_L \circ \theta_2(u_{[0,i]})$$

$$= u_{[0,i]} \qquad (5.5,8)^*$$

for all input-output pairs $(u_{[0,i]}, y_{[0,i]})$ and for all i. A left inverse exists when it is possible to uniquely determine the input sequence which produced any known output sequence. A right inverse is any operator θ_R: $Y \to U$ such that

$$\theta \circ \theta_R(y_{[0,i]}) = y_{[0,i]} \qquad (5.5,9)$$

---

*A more specific definition of the type of inverses to be discussed for (5.5,1) will be given below.

for all input-output pairs $(u_{[0,i]}, y_{[0,i]})$ and for all i. A right inverse exists when it is possible to find some input sequence $u_{[0,i]}$ for any given output sequence $y_{[0,i]}$.

Example 15

Suppose that (5.5,1) has zero initial conditions. Then its input-output map is just its transfer function matrix G(s) (2.2,4). Now, it is clear that (5.5,1) is left invertible if G(s) has normal rank m. Similarly, (5.5,1) has a right inverse if its associated transfer function matrix has normal rank r. □

As these preliminary remarks show, the problem of left and right inverses is approached from the input-output mapping point of view. Therefore, the analysis below will consider only systems (5.5,1) which are controllable and observable. It will also be assumed that

$$\mathcal{N}\begin{bmatrix} B \\ \\ D \end{bmatrix} = \mathcal{N}\begin{bmatrix} C^T \\ \\ D^T \end{bmatrix} = 0 \; ; \quad \text{i.e.} \tag{5.5,10}$$

that there are no redunant inputs and outputs. This will be assumed throughout this section.

Left Inverses

The discussion of left inverses will consider two cases: when x_0 is known and when x_0 is unknown. Suppose that x_0 is known but not equal to zero. Then, from (5.5,6),

$$\bar{y}_{[0,i]} = y_{[0,i]} - Q_{[x_0,i]}x_0, \tag{5.5,11}$$

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where $y_{[0,i]}$ is the known output sequence, is the output sequence of (5.5,1) for the same input sequence but with zero initial conditions. Hence, if x_0 is given, without loss of generality, it can be taken to be zero. This is done below.

Introduce the following definition:

Definition 15 [33]

A system (5.5,1) has an L-delay left inverse if for any non-negative integer i, the input segment $u_{[0,i]}$ can be uniquely determined by the output segment $y_{[0,i+L]}$. □

The immediate goal is to establish the conditions under which an L-delay left inverse exists for a system (5.5,1).

Lemma 5 [33]

A system (5.5,1) has an L-delay left inverse if and only if for an output segment $y_{[0,L]}$:

(a) if x_0 is given, then u_0 can be uniquely determined

(b) if x_0 is not known, then x_0 can be uniquely determined

Proof: First, assume $x_0 = 0$. Necessity follows directly from Definition 15. Conversely, suppose an output segment $y_{[0,L]}$ uniquely determines u_0. Then the effect of u_0 on the entire output sequence can be subtracted out:

$$\bar{y}_{[0,i]} = y_{[0,i]} - \begin{bmatrix} CA^{i-1}B \\ \vdots \\ CB \\ D \end{bmatrix} u_o = \begin{bmatrix} CA^{i-2}B & CA^{i-3} & \cdots & CB & D \\ \vdots & \vdots & & & \\ CB & D & & & \\ D & & & & \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} u[1,i] \quad (5.5,12)$$

where (5.5,12) is obtained from (5.5,6) with $x_0 = 0$. Observe that

$$\bar{y}_{[1,i]} = Q_{[0,i-1]} u_{[1,i]} \tag{5.5,13}$$

By hypothesis, if $i-1 \geq L$, (5.5,13) can be solved to uniquely determine u_1. Now induction establishes part (a).

Now assume that x_0 is unknown and that (5.5,1) has an L-delay left inverse. Then $u_{[0,i]}$ can be uniquely determined from the output segment $y_{[0,i+L]}$ by Definition 15. In particular, $u_{[0,n]}$ can be found. Then (5.5,6) yields

$$y_{[0,n]} - Q_{[0,n]} u_{[0,n]} = Q_{[x_0,n]} x_0 \tag{5.5,14}$$

But $Q_{[x_0,n]}$ *is the observability matrix*. (See (5.5,4)). Since (5.5,1) is assumed to be observable, $\rho[Q_{[x_0,n]}] = n$. This implies (5.5,14) can be solved uniquely for x_0.

Suppose x_0 is uniquely determined by $y_{[0,L]}$. Then the effect of x_0 on $y_{[0,L]}$ can be subtracted off as noted in (5.5,11). This gives the equation

$$y_0 - Cx_0 = Du_0 \tag{5.5,15}$$

From (5.5,11) the following relationship is obtained by dropping the blocks associated with the first time unit:

$$\bar{y}_{[1,i]} = Q_{[0,i-1]} u_{[1,i]} + Q_{[Bu_0,i-1]} Bu_0 \tag{5.5,16}$$

so that $\bar{y}_{[1,i]}$ is the output sequence to (5.5,1) on $[1,\infty)$ with an input sequence $u_{[1,i]}$ and initial condition $x_1 = Bu_0$. By hypothesis, x_1 can be

determined from the segment $\bar{y}_{[1,L+1]}$. By assumption (5.5,10), it follows that $x_1 = Bu_0$ and (5.5,15) taken together uniquely determine u_0. An induction argument completes the proof. □

The following theorem gives conditions for the existence of an L-delay left inverse in terms of Algorithm I (Section 5.2). Recall that $\mathcal{L}_i$ is the i-th unknown-input unobservable subspace, Definition 11. Furthermore, throughout the rest of this section, F_i shall represent the matrix F_i defined in Algorithm 1, (5.2,15).

Theorem 18

a) If x_0 is unknown, an L-delay left inverse for a system (5.5,1) exists if and only if $\mathcal{L}_{L+1} = 0$.

b) If x_0 is known, an L-delay left inverse for a system (5.5,1) exists if and only if $\rho[F_{L+1}] = m$.

Proof: By equation (5.5,6)

$$y_{[0,i]} = Q_i \begin{bmatrix} x_0 \\ u_{[0,i]} \end{bmatrix} \tag{5.5,17}$$

Compare (5.5,17) to equation (5.2,17). They are exactly the same if the row of blocks

$$[M_0A^i, M_0A^{i-1}B, \ldots, M_0AB, M_0B] \tag{5.5,18}$$

is attached to the top of Q_i and M_0x_{i+1} is attached to the top of the vector $y_{[0,i]}$. If the proof of Theorem 10, where (5.2,17) appears, (5.2,17)

is reduced to (5.2,20) under the assumption $M_0 = 0$. It follows that Q_i can also be reduced to (5.2,20); i.e. there exists a non-singular matrix S such that

$$SQ_i = \bar{Q}_i = \begin{bmatrix} G_1A^{i-1} & G_1A^{i-2}B & \cdots & G_1B & F_1 \\ G_2A^{i-2} & G_2A^{i-3}B & \cdots & F_2 & 0 \\ \vdots & \vdots & & & \vdots \\ G_{i+1} & F_{i+1} & & & \vdots \\ M_{i+1} & 0 & \cdots & \cdots & 0 \end{bmatrix} = [\bar{Q}_{[x_0,i]}\bar{Q}_{[0,i]}] \tag{5.5,19}$$

where the matrices G_i, F_i, M_i are produced by Algorithm 1.

To prove part (a), assume that x_0 is unknown. First, suppose (5.5,1) has an L-delay left inverse. If x_0 can be uniquely determined for all x_0, it follows from (5.5,19) that $\mathcal{N}[M_{L+1}] = 0$. By Theorem 10, $\mathcal{L}_{L+1} = \mathcal{N}[M_{L+1}]$. Conversely, suppose $\mathcal{L}_{L+1} = 0$. This implies $\mathcal{N}[M_{L+1}] = 0$. Then (5.5,19) shows that x_0 can be uniquely determined from $y_{[0,L]}$. Hence, by Lemma 5, an L-delay left inverse for system (5.5,1) exists.

To show part (b), assume that $x_0 = 0$. Then equations (5.5,17) and (5.5,19) reduce to

$$Sy_{[0,L]} = \bar{Q}_{[0,L]}u_{[0,L]} \tag{5.5,20}$$

Suppose that F_{L+1} has rank m, i.e. F_{L+1} is a square, non-singular matrix. Then u_0 is uniquely determined. By Lemma 5, the system has a L-delay left inverse.

Now suppose that system (5.5,1) has an L-delay left inverse but F_{L+1} doesn't have full rank. Note that

$$\mathcal{N}[F_i] \supset \mathcal{N}[F_{i+1}] \qquad (5.5,21)$$

(This is easily seen from the structure of columns that the F_i's are associated with). Hence, if $\mathcal{N}[F_{L+1}] \neq 0$, then $\mathcal{N}[F_i] \neq 0$, $i = 1,2,\ldots,L$. It follows that u_0 can't always be uniquely determined, contradicting Lemma 5. □

Part (b) of Theorem 18 was first proved by Silverman and Payne [34].

A number of known results follow immediately from Theorem 18.

Corollary 2

The following statements are true for a system with unknown initial conditions:

(a) There exists a least integer q_o such that if the system (5.5,1) has a q-delay left inverse and a q-delay left inverse for every $q \geq q_o$.

(b) $q_o \leq n$ if q_o exists.

(c) The following statements are equivalent for the system (5.5,1):

(1) (5.5,1) has a q-delay left inverse.

(2) $\mathcal{L}^* = 0$

(3) There is no x_0 and no non-zero input sequence $u_{[0,n]}$ followed by zero inputs such that the output sequence $y_{[0,i]}$ is identically zero for all i.

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Proof: To see part (a), recall that $d(\mathcal{L}_i)$ is monotonically nonincreasing (Lemma 1, Section 5.2). If $d(\mathcal{L}_q) = 0$ for some q, then there exists a first integer, q_o, such that $d(\mathcal{L}_{q_o}) = 0$ and $d(\mathcal{L}_q) = 0$ for all $q \geq q_o$. The result follows by Theorem 18.

Part (b) is also a consequence of Lemma 1 and Theorem 18. Since $\mathcal{L}_n = \mathcal{L}_{n+j}$, $j = 1,2,\ldots$ (by Lemma 1), if $\mathcal{L}_n \neq 0$, then the system (5.5,1) doesn't have a left inverse (Theorem 18). Hence, if a q-delay left inverse exists for a system (5.5,1), then $q_o \leq n$ by part (a).

In part (c), the equivalence of (1) and (2) is a consequence of parts (a) and (b) and Theorem 12 which states that $\mathcal{L}^* = \mathcal{L}_n$. From Definition 11, it is seen that (3) simply states that $\mathcal{L}_n = 0$. By Theorem 12, this is equivalent to $\mathcal{L}^* = 0$. □

The corollary characterizes to what extent past values of x_i affect the present value of the output. The existence condition in part (c) was first established by Bengtsson [2] for the case $D = 0$.

Corollary 3

The follows statement are true for a system (5.5,1) with known initial conditions.

(a) If a system has an L-delay left inverse, then there exists a least integer L_0 such that the sytem has a L_0-delay left inverse and an L-delay left inverse for every $L \geq L_0$.

(b) $L_0 \leq n$ if L_0 exists.

(c) The following statements are equivalent.

(1) A system has a L-delay left inverse.

(2) $\rho[Q_{[0,L]}] - \rho[Q_{[0,L-1]}] = m$. Furthermore, L_0 is the first integer for which this is true.

(3) There exists no input segment $u_{[0,n]}$, not equal to zero, followed by all zeros which produces an identically zero output sequence.

(4) $R^* = 0$.

Proof: The properties of this corollary follow from the properties of the matrix $Q_{[0,L]}$ and its reduced form $\overline{Q}_{[0,L]}$ defined in (5.5,19).

Part (a) follows from the fact that the first column of blocks of $M_{[0,L]}$ has a monotonically non-decreasing column rank. If $\rho[F_{L_0+1}] = m$ (the column rank of the first column of blocks), then $\rho[F_L] = m$ and $L \geq L_0+1$. Theorem 18 completes the proof of part (a).

The Cayley - Hamilton theorem applied to the first column of blocks proves part (b).

To prove part (c) consider the matrix $\overline{Q}_{[0,L]}$. This matrix shows the rank of both $Q_{[0,L]}$ and $Q_{[0,L-1]}$; i.e.

$$\begin{aligned} \rho[Q_{[0,L]}] &= \sum_{i=1}^{L+1} \rho[F_i] \\ \rho[Q_{[0,L-1]}] &= \sum_{i=1}^{L} \rho[F_i] \end{aligned} \tag{5.5,22}$$

so that

$$\rho[Q_{[0,L]}] - \rho[Q_{[0,L-1]}] = \rho[F_{L+1}] \tag{5.5,23}$$

Now Theorem 18 establishes the equivalence of (1) and (2). It is also

clear that the last remark follows from part (a).

To show the equivalent of (1) and (3), suppose the system is left invertible. Then, by Theorem 18 and part (b), F_L is a square non-singular matrix for some $L \leq n$. Hence, a non-zero u_o will produce a non-zero output. Conversely, suppose (3) but the system is not invertible. Then it follows that $\rho[F_i] < m$, $i = 0,1,\ldots,n$. Then, it is possible to find a non-zero input segment $u_{[0,n]}$ followed by all zeros which would produce an identically zero output. This contradiction establishes the equivalence of (1) and (3).

That (3) implies (4) follows from the fact that $\mathcal{R}^* \subset \mathcal{R}_n$ (See equation (5.2,30)). From Definition 13, it follows that (3) implies $\mathcal{R}_n = 0$. Hence, $\mathcal{R}^* = 0$. Now assume that $\mathcal{R}^* = 0$ but that the system has no L-delay left inverse. By (3), there exists a non-zero input segment $\bar{u}_{[0,n]}$ followed by all zeros which produces an identically zero output. Clearly,

$$\bar{u}_i \in \mathcal{N}[D] \quad i = 0,1,\ldots,n \tag{5.5,24}$$

It follows that

$$\bar{u}_i \notin \mathcal{N}[B] \quad i = 0,1,\ldots,n \tag{5.5,25}$$

since

$$\mathcal{N}[B] \cap \mathcal{N}[D] = 0 \tag{5.5,26}$$

by assumption. Therefore, this input segment generates a state sequence $\bar{x}_i$, $i = 1,\ldots,n+1$. By Definition 11, these $\bar{x}_i$ are elements of $\mathcal{L}_n$. For if $x_0 = \bar{x}_i$, then the control sequence

$$u_j = \begin{cases} \bar{u}_{i+j} & 0 \le j \le n-i \\ 0 & n-i < j \le n \end{cases} \tag{5.5,27}$$

will produce an identically zero output. Since

$$x_1 = B\bar{u}_0 \ne 0 \tag{5.5,28}$$

it follows that $\mathcal{L}_n \ne 0$. But, by construction, x_1 is also an element of $\mathcal{R}_n$, Definition 13. The input segment

$$u_j = \begin{cases} 0 & 0 \le j \le n-2 \\ \bar{u}_0 & j = n-1 \end{cases} \tag{5.5,29}$$

shows this. Now, by Theorem 13,

$$\mathcal{R}^* = \mathcal{R}_n \cap \mathcal{L}_n \ne \phi \tag{5.5,30}$$

This contradiction shows that (4) implies (1). □

Comparison of Corollary 3 with Corollary 4 shows that there are similar results for systems with known and unknown initial conditions. The main difference is that if the initial condition is unknown, it is necessary to reconstruct the state, where as if the initial condition is known this is not necessary. This leads to the tighter restrictions on systems with unknown initial conditions. ($\mathcal{L}^* = 0$ implies $\mathcal{R}^* = 0$, but not conversely).

The integer L_0 defined in part (a) of Corollary 3 is called the <u>inherent integration</u> of the system [35]. It should be noted that this number, L_0, is <u>not</u> the same as the integer q_o that was introduced in part (a)

of Corollary 2. Silverman and Payne have shown that $L_0 \leq q_o$ [34] but this is all that can be claimed for left invertible systems. Sain and Massey have shown [35] that if G(s) is a transfer function matrix for a left invertible system with inherent integration L_0 and $\hat{G}(s)$ is the transfer function matrix for that left inverse, then

$$\hat{G}(s)G(s) = \frac{I_m}{s^{L_o}} \qquad (5.5,31)$$

The number L_0 characterizes the extent to which past values of the control affect the present value of the output.

Part (c) of Corollary 3 gives 3 equivalent characterizations of systems with left inverses. Sain and Massey [35] proved (2) and (3); Wonham and Morse [36] proved (4) for the case $D = 0$. Corollary 3 and Theorem 18 together show the equivalence of the characterizations of left inverse systems given by Sain and Massey [35] and Silverman and Payne [34]. This equivalence was also noted in [37] and [38].

<u>Right Inverses</u>

The right inversion problem for (5.5,1) can be analyzed in much the same way as left inverses were analyzed above. The right inversion problem occurs when it is desired to find an input sequence for (5.5,1) which will produce some specified output sequence. Note that this problem is slightly different in that the specified output sequence is not known to be in the image of the operator ∂ (5.5,7). This leads to some differences in the way the initial condition, x_0, is handled. Three cases are distinguished: 1) x_0 is unknown, 2) x_0 is known and fixed, and 3) x_0 is known, but can be chosen to match the specified output sequence. To motivate the

rest of the discussion of initial conditions, consider the following definition:

Definition 16

A system (5.5,1) has an L-delay right inverse if for every non-negative integer i, an input segment $u_{[0,i]}$, which produces the given output segment $y_{[0,i]}$, can be determined from the output segment $y_{[0,i+L]}$ □

Definition 16 differs from Definition 15 in two ways. First, the input sequence in Definition 16 is not required to be unique. Secondly, Definition 16 implies that the output sequence $y_{[0,i]}$ is in the image of the operator θ. Since θ is parameterized by x_0, the class of output sequences $y_{[0,i]}$ for which it is possible to obtain an input sequence $u_{[0,i]}$ is dependent on x_0. To see this explicitly, recall the matrices Q_i (5.5,4) and $\bar{Q}_i$ (5.5,19), which relate the input and output sequences as follows:

$$\bar{y}_{[0,i]} = Sy_{[0,i]} = SQ_i \begin{bmatrix} x_0 \\ \\ u_{[0,i]} \end{bmatrix} = \bar{Q}_{[x_0,i]}x_0 + \bar{Q}_{[0,i]}u_{[0,i]} \tag{5.5,32}$$

where S is defined in (5.5,19). Consider again the three cases listed for the initial condition x_0. First, suppose that the initial condition is unknown. Then x_0 must be determined if (5.5,32) is to be solved for $u_{[0,i]}$. This implies that $\mathcal{N}[M_{i+1}] = 0$ where M_{i+1} is defined in (5.5,19). This is quite a strong condition but further treatment of the case is deferred until later. Note that if x_0 can be determined, then the calcula-

tion of a suitable $u_{[0,i]}$ depends on the matrix $\overline{Q}_{[0,i]}$.

Secondly, suppose that x_0 is known and can be chosen arbitrarily. Then from the bottom row of blocks in (5.5,32) (see (5.5,19))

$$\tilde{y}_{[0,i]} = M_{i+1}x_0 \qquad (5.5,33)$$

must be solvable for some x_0 where $\tilde{y}_{[0,i]}$ is the appropriate subvector of $\overline{y}_{[0,i]}$. This implies that $\tilde{y}_{[0,i]} \in \mathcal{R}[M_{i+1}]$ which puts a constraint on the class of sequences $y_{[0,i]}$ for which (5.5,32) is solvable. Again, if (5.5,33) is satisfied then the question of the existence of a right inverse rests on the matrix $\overline{Q}_{[0,i]}$.

Finally, suppose x_0 is known and fixed. Then (5.5,33) uniquely specifies $\tilde{y}_{[0,i]}$ and all output sequences $y_{[0,i]}$ must satisfy (5.5,33) to be in the image of θ. If this requirement is met, the existence of a suitable input sequence $u_{[0,i]}$ depends on $\overline{Q}_{[0,i]}$ as above.

Two remarks are in order. First, the condition in (5.5,33) can be decomposed into constraints on certain subvectors of each output vector y_i [34] but the details will not be given here. Secondly, suppose that $r \leq m$ in (5.5,1) and that $\rho[D] = r$. Then the application of Algorithm I to this system immediately yields $M_i = 0$ for all i. In this case, the constrait (5.5,33) doesn't exist and an appropriate input sequence $u_{[0,i]}$ can be found for every output sequence $y_{[0,i]}$ (as will be shown below).

Now suppose that the proposed output sequence meets the requirements imposed by the initial condition, and that the effect of the initial condition has been subtracted out, as in (5.5,11). The possible existence of an appropriate input sequence $u_{[0,i]}$ is still to be determined. This

question is answered by the following lemma and theorem.

Lemma 6

A system (5.5,1) has an L-delay right inverse if and only if u_0 can be determined from $y_{[0,L]}$.

Proof: Exactly the same as Lemma 5, part (a). □

The following theorem characterizes L-delay right inverses.

Theorem 19

(a) A system (5.5,1) has an L-delay right inverse if and only if $\rho[F_{L+1}] = r$.

(b) If a system has an L-delay right inverse, then there exists *a least integer* L_0 *such that the system has an* L_0*-delay right inverse and* an L-delay right inverse for every $L \geq L_0$.

(c) If L_0 exists then $L_0 \leq n$.

(d) The system (5.5,1) has an L-delay right inverse if and only if $\rho[Q_{[0,L]} - \rho[Q_{[0,L-1]}] = r$. Furthermore, L_0 is the least integer for which this is true.

Proof: This theorem follows from the equation

$$\hat{y}_{[0,L]} = S(y_{[0,L]} - Q_{[x_0,L]}x_0) = SQ_{[0,L]}u_{[0,L]} = \bar{Q}_{[0,L]}u_{[0,L]} \tag{5.5,34}$$

where S and $\bar{Q}_{[0,L]}$ are defined in (5.5,19). Note that the matrices F_i appear in $\bar{Q}_{[0,L]}$.

To show part (a), suppose that $\rho[F_{L+1}] = r$ for some L. Then (5.5,34) shows that it is possible to determine a u_0. By Lemma 6, the

system has a L-delay right inverse. Now suppose the system has an L-delay right inverse but $\rho[F_{L+1}] \neq r$. Then from (5.5,34), it is easy to see that u_0 can't be determined for every permissible output sequence. Hence by Lemma 6, a L-delay right inverse doesn't exist. This contradiction establishes part (a).

Part (b) follows by noting that the row rank of the top row of blocks in $Q_{[0,L]}$ is non-decreasing. The arguement is similar to the one used to prove part (a) of Corollary 3.

Part (c) is established by applying the Cayley-Hamilton theorem to the top row of blocks of $Q_{[0,L]}$ and using part (b).

Part (d) follows directly from part (a). The details are similar to the proof of (2) of part (c) in Corollary 3. □

The criteria for the existence of a right inverse in part (a) was first presented by Silverman and Payne [34]; the criteria in parts (b)-(d) by Sain and Massey [35]. Definition 16 for a system with a known initial condition was first introduced in [39] and called <u>functional reproducibility</u>.

Consider again the question of a right inverse for a system (5.5,1) with unknown initial conditions. Suppose that such a system does have a L-delay right inverse. By the remarks proceeding Lemma 6, it follows that $\mathcal{N}[M_{L+1}] = 0$. By Theorem 10, this implies $\mathcal{L}_{L+1} = 0$. Now by Theorem 18, this system also has a L-delay left inverse. A standard result says that in this case the inverse is unique and the distinction between right and left need not be made.

There is a duality between left and right inverses. Consider

the dual system of (5.5,1)

$$x_{i+1} = A^T x_i + C^T u_i$$
$$y_i = B^T x_i + D^T u_i \qquad (5.5,35)$$

If $x_0 = 0$, the solution to this system is given by

$$y_{[0,i]} = Q^T_{[0,L]} u_{[0,L]} \qquad (5.5,36)$$

where $Q_{[0,L]}$ is defined in (5.5,4). Now the duality between Corollary 3 and Theorem 19 is completely obvious.

Theorem 20 [35]

If the initial condition of (5.5,1) is zero, then the system (5.5,1) has an L-delay left inverse if and only if its dual system (5.5,35) has an L-delay right inverse. □

Example 16

In Example 15, it was pointed out that a transfer function matrix G(s) has a left inverse when its normal rank in m. In this case, it is clear that the dual system $G^T(s)$ has a right inverse.

For completeness, it is noted that if the system (5.5,1) has zero initial conditions, then its transfer function matrix G(s) has normal rank $\rho[F_{n+1}]$. See [34] for details. □

Minimal Order Left Inverses

It has already been shown that for transfer function matrices, the transmission zeros of a transfer function matrix are the poles of an inverse transfer function matrix (Section 3.4). In what follows, the left inverse of a system (5.5,1) will be constructed to show that this result

generalizes to state space systems. It will be assumed that the left inverse system will have the form

$$\begin{aligned} \omega_{i+1} &= \hat{A}\,\omega_i + N_1(p)y_i \\ u_i &= \hat{C}\,\omega_i + N_2(p)y_i \end{aligned} \tag{5.5,37}$$

where $N_1(s)$ and $N_2(s)$ are polynomial matrices and p is the unit delay. A <u>minimal left inverse</u> is any left inverse with representation (5.5,37) such that ω_i has minimal dimension [2].

Bengtsson [2] has given a construction of a minimal left inverse for systems with $D = 0$ and unknown initial conditions. To study systems with $D \neq 0$, it is necessary to further delineate the structure of the D matrix. By applying input and output transformations, the system (5.5,1) can be reduced to

$$x_{i+1} = Ax_i + [B_1\ B_2]\begin{bmatrix} \bar{u}_i \\ \hat{u}_i \end{bmatrix} \tag{5.5,38a}$$

$$\begin{bmatrix} \bar{y}_i \\ \hat{y}_i \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x_i + \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_i \\ \hat{u}_i \end{bmatrix} \tag{5.5,38b}$$

where D_1 is a square non-singular matrix. Let C_2 be a $r_1 \times n$ matrix. The following three lemmas will be needed in the construction of a left inverse for (5.5,38).

Lemma 7

If $\mathcal{L}^* = 0$ for the system (5.5,38), then $\mathcal{L}^* = 0$ for the system $(A, B_2, C_2, 0)$.

Proof: Suppose that $\mathcal{L}^* = 0$ for (5.5,38) but $\mathcal{L}^* \neq 0$ for $(A, B_2, C_2, 0)$. By Theorem 12, $\mathcal{L}^* = \mathcal{L}_n$. If $\mathcal{L}_n \neq 0$, there exists an $x_0 \neq 0$ and an input segment $\hat{u}'_{[0,n-1]}$ such that the output segment $y_{[0,n-1]}$ is identically zero (Definition 11). This input segment generates as state segment x'_i, $i = 1, \ldots, n$. Now consider the full system (5.5,38) with x'_0. The equation

$$\overline{y}_0 = 0 = C_1 x'_0 + D_1 \overline{u}_0 \tag{5.5,39}$$

can be solved for $\overline{u}_0$ since D_1 is square and non-singular. Furthermore, the equation

$$x'_1 = Ax'_0 + B_2\hat{u}'_0 = Ax'_0 + B_1\overline{u}_0 + B_2\hat{u}_0 \tag{5.5,40}$$

yields

$$B_2\hat{u}_0 = B_2\hat{u}^1 - B_1\overline{u}_0 \tag{5.5,41}$$

By assumption (5.5,10), the special structure of (5.5,38) shows that $\mathcal{N}[B_2] = 0$. It follows that B_2 has a left inverse B_2^+ so that (5.5,41) can be solved for $\hat{u}_0$; i.e.

$$\hat{u}_0 = \hat{u}'_0 - B_2^+ B_1 \overline{u}_0 \tag{5.5,42}$$

Let

$$v_0 = \begin{bmatrix} \bar{u}_0 \\ \\ \hat{u}_0 \end{bmatrix} \tag{5.5,43}$$

where $\bar{u}_0$ is obtained from (5.5,39) and $\hat{u}_0$ is given by (5.5,42). By induction, this procedure will generate an input segment $v_{[0,n-1]}$ which produces the state segment x_i', $i = 1,\ldots,n$, starting from x_0', such that the output is identically zero. (Note that $\hat{y}_i = C_2 x_i' = 0$ by choice of x_i'. Equation (5.5,39) shows that $\bar{y}_i$ is zero). Hence, by Definition 11, $x_0' \in \mathcal{L}_n \neq 0$, where $\mathcal{L}_n$ is defined for the full system (A,B,C,D). By Theorem 12, this is a contradiction as $\mathcal{L}^* = 0$ by hypothesis. □

Note that this proof actually shows that $\mathcal{L}_i$ for the system $(A,B_2,C_2,0)$ is the same as $\mathcal{L}_i$ for the full system (5.5,38).

<u>Lemma 8</u>

If $\mathcal{L}^* = 0$ for the system (5.5,38), then there exists $n \times r_1$ matrices N_i such that

$$\Sigma_{k=0}^{q_o} N_{q_o - k} C_2 A^k = I_n$$

$$\Sigma_{k=i}^{q_o} N_k C_2 A^{i-k} B_2 = 0 \qquad i = 0,1,2,\ldots,q_o$$

where q_o is defined in Corollary 2.

Proof: Since $\mathcal{L}^* = 0$ for the system (5.5,38), $\mathcal{L}_{q_o+1} = 0$ as was shown in the proof of Corollary 2. By the remark following the proof of Lemma 7, $\mathcal{L}_{q_o+1} = 0$ for the system $(A,B_2,C_2,0)$. Consider the matrix Q_{q_o} (5.5,4) defined for $(A,B_2,C_2,0)$. This matrix can be reduced to $\bar{Q}_{q_o}$ as was done in the proof of Theorem 18 (5.5,17-19). By Theorem 10, $\mathcal{N}[M_{q_o+1}] = 0$ where M_{q_o+1} occurs in $\bar{Q}_{q_o}$. From the structure of $\bar{Q}_{q_o}$, it is seen that $\mathcal{N}[M_{q_o+1}] = 0$

implies that the columns of $Q_{[x_0,q_0]}$ are linearly independent from the columns of $Q_{[0,q_0]}$. This implies that there exists a $n \times (q_0+1)r_1$ matrix N such that

$$NQ_{[x_0,q_0]} = I_n$$
$$NQ_{[0,q_0]} = 0 \tag{5.5,44}$$

Partitioning N compatibly with $Q_{[x_0,q_0]}$, i.e.

$$N = [N_0, N_1 \dots N_{q_0}] \tag{5.5,45}$$

gives the desired result. □

Lemma 9

If a system (5.5,38), with unknown initial condition x_0, has a q_0-delay left inverse, then there exists a polynomial matrix N(s) such that for all $\hat{u}_{[0,i]}$, $\hat{y}_{[0,i]}$, and all x_0

$$x_i = N(p)\hat{y}_i$$

$$\hat{u}_i = B_2^+(pI-A)N(p)\hat{y}_i$$

where p is a unit delay, $\hat{u}_i$, $\hat{y}_i$, and B_2 are defined by (5.5,38) and B_2^+ is a left inverse of B_2.

Proof: If the system (5.5,38) has a q_0-delay left inverse, then, by Corollary 2, $\mathcal{L}_{q_0+1} = 0$. Hence, the matrices N_i defined in Lemma 8 exist. Consider the system $(A,B_2,C_2,0)$.

$$N_{q_0}\hat{y}_i = N_{q_0}C_2x_i$$

$$N_{q_0-1}\hat{y}_{i+1} = N_{q_0-1}(C_2Ax_i + C_2B_2u_i) \tag{5.5,46}$$

$$= N_{q_0-1}C_2Ax_i$$

by Lemma 8. This gives

$$N_{q_0}\hat{y}_i + N_{q_0-1}\hat{y}_{i+1} = (N_{q_0} + pN_{q_0-1})\hat{y}_i = (N_{q_0}C_2 + N_{q_0-1}C_2A)x_i \tag{5.5,47}$$

If this procedure is continued for q_0+1 steps, it is found that

$$(\Sigma_{k=0}^{q_0} p^{q_0-k} N_k)\hat{y}_i = N(p)\hat{y}_i = (\Sigma_{k=0}^{q_0} N_{q_0-k}CA^k)x_i = x_i \tag{5.5,48}$$

again by Lemma 8. Now, from the system $(A,B_2,C_2,0)$

$$(pI-A)x_i = B_2\hat{u}_i \tag{5.5,49}$$

Substituting (5.5,48) into (5.5,49) yields

$$\hat{u}_i = B_2^+ \ (pI-A)N(p)\hat{y}_i \tag{5.5,50}$$

where B_2^+ is a left inverse of B_2. Note that B_2^+ exists as $\mathcal{N}[B_2] = 0$ from the requirement $\mathcal{N}[B] \cap \mathcal{N}[D] = 0$ for the system (5.5,38). Equations (5.5,48) and (5.5;50) are the required results, as (5.5,46-50) hold for all $i \geq 0$. □

From Lemma 9, the complete left inverse of (5.5,38) is easily constructed. From (5.5,38b)

$$\bar{y}_i = C_1x_i + D_1\bar{u}_i \tag{5.5,51}$$

or

$$\bar{u}_i = D_1^{-1}\bar{y}_i - D_1^{-1}C_1x_i \tag{5.5,52}$$

since D_1 is square and non-singular. Combining equations (5.5,50) and (5.5,52), it follows that

$$u_i = \begin{bmatrix} \bar{u}_i \\ \hat{u}_i \end{bmatrix} = \begin{bmatrix} D_1^{-1} & -D_1^{-1}C_1N(p) \\ 0 & B_2^{+}(pI-A)N(p) \end{bmatrix} \begin{bmatrix} \bar{y}_i \\ \hat{y}_i \end{bmatrix} \tag{5.5,53}$$

$$= \bar{N}(p)\,y_i$$

Equation (5.5,53) holds for all $i \geq 0$. Hence, (5.5,53) combined with (5.5,48) constitute a left inverse for (5.5,38). The following theorem has been proved:

<u>Theorem 21</u>

If a system (5.5,38) with unknown initial condition x_0 has a q_0-delay left inverse, then there exists a polynomial matrix $N(s)$, defined by Lemma 8, and a second polynomial matrix $\bar{N}(s)$, defined by (5.5,53) such that

$$\begin{aligned} x_i &= N(p)\hat{y}_i = [0 \quad N(p)]y_i \\ u_i &= \bar{N}(p)y_i \end{aligned} \tag{5.5,54}$$

where p is the unit delay, holds for all input and output sequences and all x_0; i.e. (5.5,54) is a left inverse for (5.5,38). □

Note that this left inverse is minimal as the dimension of ω_i in (5.5,37) is zero.

This construction was given by Bengtsson [2] for systems with $D \equiv 0$. Lemma 8 also appeared there for the special case $q_0 = n$. The left inverse system (5.5,54) can be interpreted as a bank of delays, $N(p)$, followed by a dynamical system. From the construction of $N(p)$ given in the proof of Lemma's 8 and 9, it is seen that q_0 is the minimum number of delays needed to implement this inverse system. Hence, the construction of the left inverse given here is for a larger class of systems and it also identifies the maximum order of a polynomial in $N(p)$; i.e. the maximum number of delays needed to implement the left inverse system.

Now consider a system (5.5,38) with zero initial conditions. If this system has a left inverse, by Corollary 3 $\mathcal{R}^* = 0$. By Theorem 11, there exists a feedback matrix K such that (5.5,38) can be placed in the following canonic form:

$$x_{i+1} = (A+BK)x_i + Bv_i = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \tilde{x}_i \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \bar{v}_i \\ \hat{v}_i \end{bmatrix} \tag{5.5,55a}$$

$$y_i = (C+DK)x_i + Dv_i = \begin{bmatrix} \bar{y}_i \\ \hat{y}_i \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{21} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \tilde{x}_i \end{bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{v}_i \\ \hat{v}_i \end{bmatrix} \tag{5.5,55b}$$

$$u_i = Kx_i + v_i = K_1\bar{x}_i + K_2\tilde{x}_i + v_i \tag{5.5,55c}$$

In general, this transformation will require a change of basis in state space. Without loss of generality, it will be assumed that this has been

done. In comparing (5.5,55) to canonic form (5.2,24), note that $\mathcal{R}^* = 0$ for (5.5,55) implies that $B_4 = 0$ where B_4 is defined in (5.2,24). This implies that $B_1 = [B_{11} \;\; B_{12}]$ and $B_2 = [B_{21} \;\; B_{22}]$ where B_1 and B_2 are defined in (5.2,24). Of course, D_1 is a square non-singular matrix defined in (5.5,38).

The first step in constructing a left inverse for the system (5.5,55) is defined in the following lemma:

Lemma 10

If the system (5.5,55) has a left inverse when $x_0 = 0$, there exist polynomial matrices $N(p)$ and $\overline{N}(p)$ such that

$$\overline{x}_i = N(p)\hat{y}_i$$

$$v_i = \begin{bmatrix} \overline{v}_i \\ \\ \hat{v}_i \end{bmatrix} = \begin{bmatrix} D_1^{-1} & -D_1^{-1}C_{11}N(p) \\ \\ 0 & B_{12}^{+}(pI-A)N(p) \end{bmatrix} \begin{bmatrix} \overline{y}_i \\ \\ \hat{y}_i \end{bmatrix}$$

$$= \overline{N}(p)y_i$$

where B_{12}^{+} is a left inverse of B_{12} and p is a unit delay.

Proof: The input-output operator of (5.5,55) is found by neglecting the unobservable part of (5.5,55); i.e. it corresponds to the input-output operator of the subsystem $(A_{11}, [B_{11} \;\; B_{12}], C_{11}, [D_1 \;\; 0])$. Since $\mathcal{L}^*$ is maximal for (5.5,55), it follows that $\mathcal{L}^*$ for $(A_{11}, [B_{11} \;\; B_{12}], C_{11}, [D_1 \;\; 0])$ is zero. Now, if it can be shown that a left inverse exists for B_{12}, then Theorem 21 can be applied to $(A_{11}, [B_{11} \;\; B_{12}], C_{11}, [D_1 \;\; 0])$ to give the desired result. To show the existence of a left inverse for

B_{12}, it will be shown that $\mathcal{N}[B_{12}] = 0$. Suppose that $\mathcal{N}[B_{12}] \neq 0$. Since $\mathcal{N}[B] \cap \mathcal{N}[D] = 0$ by assumption, it follows that $\mathcal{N}[B_{12}] \cap \mathcal{N}[B_{22}] = 0$. Chose a non-zero input sequence $u_{[0,i]}$ such that $u_j \in \mathcal{N}[B_{12}]$ for all j. This input sequence generates a non-zero state sequence x_i since $u_j \notin \mathcal{N}[B_{22}]$ for all j. The structure of (5.5,55) shows that $\bar{x}_j = 0$ for all j and that the output is identically zero. By Definition 13, it follows that $\mathcal{R}^* \neq 0$ for (5.5,55), a contradiction. Thus, $\mathcal{N}[B_{12}] = 0$, B_{12} has a left inverse, and the lemma is proved. □

Before a minimal left inverse for (5.5,55) can be given, the following two results are needed.

Lemma 11 [2]

The pair (K_2, A_{22}) is completely observable where K_2 and A_{22} are defined as in (5.5,55).

Proof: If the pair (K_2, A_{22}) is not completely observable, there exists a A_{22}-invariant subspace $\mathcal{V}$ in $\mathcal{N}[K_2]$. If W is a basis matrix for $\mathcal{V}$, this implies that

$$A_{22}W = WQ \tag{5.5,56}$$

for some matrix Q and

$$K_2 W = 0 \tag{5.5,57}$$

Define

$$\bar{W} = \begin{bmatrix} 0 \\ \\ W \end{bmatrix} \tag{5.5,58}$$

so that $\overline{W}$ has n rows. From the special structure of (A+BK) in (5.5,55), it follows that

$$\begin{aligned}(A+BK)\overline{W} &= \overline{W}Q \\ &= A\overline{W} + BK\overline{W} \\ &= A\overline{W} + BK_2W \\ &= A\overline{W}\end{aligned} \tag{5.5,59}$$

From the first and last lines of (5.5,59)

$$A\overline{W} = \overline{W}Q \tag{5.5,60}$$

which shows that $\mathcal{V}$ is A-invariant. It also follows that

$$\begin{aligned}(C+DK)\overline{W} &= 0 \\ &= C\overline{W} + DK\overline{W} \\ &= C\overline{W} + DK_2W \\ &= C\overline{W}\end{aligned} \tag{5.5,61}$$

From the first and last lines of (5.5,61)

$$C\overline{W} = 0 \tag{5.5,62}$$

which shows that $\mathcal{V}$ is in the null space of C. It follows that $\mathcal{V}$ is an unobservable subspace of (A,B,C,D) which contradicts the observability assumption on (C,A). Hence, (K_2,A_{22}) is a completely observable pair. □

<u>Theorem 22</u> [2]

Denote the characteristic polynomial of A_{22} by $\alpha_L(s)$. For any arbitrary left inverse of (5.5,55) with $x_0 = 0$, let $\hat{\alpha}(s)$ be the characteristic polynomial of $\hat{A}$ in representation (5.5,37). Then $\alpha_L(s)$ divides $\hat{\alpha}(s)$.

Proof: Let $\xi \in \mathcal{L}^*$ of the system in (5.5,55). Since this system is completely controllable, there exists an input segment $u_{[0,k-1]}$ such that $\xi = x_k$ for some finite k. Consider the input

$$u_i = \begin{cases} u_i & 0 \le i \le k-1 \\ Kx_i & k \le i < \infty \end{cases} \tag{5.5,63}$$

where K is the feedback matrix in (5.5,55). For $i \ge k$, the solution of (5.5,55) is given by

$$\begin{aligned} x_{i+1} &= (A + BK)x_i \\ y_i &= (C + DK)x_i \\ u_i &= Kx_i \end{aligned} \tag{5.5,64}$$

with the initial condition

$$x_k = \xi = \begin{bmatrix} 0 \\ \\ \tilde{x}_k \end{bmatrix} \tag{5.5,65}$$

since $\xi \in \mathcal{L}^*$. The special form of the system given by (5.5,55) shows that for $i \ge k$

$$u_i = K_2 A_{22}^{i-k} \tilde{x}_k$$
$$y_i = 0 \tag{5.5,66}$$

Now this u_i must be produced as an output from any left inverse of (5.5,55) with y_i as an input. Let this inverse have a representation (5.5,37) with $\hat{A}$ and $\hat{C}$ and let $\bar{A}$ and $\bar{C}$ be the observable subsystem of $(\hat{A},\hat{C})$. Since $y_i = 0$ for $i \geq k$, from (5.5,37) it follows that

$$u_i = \bar{C}\,\bar{A}^{i-k} w_k \tag{5.5,67}$$

Now $\xi = x_k$ was an arbitrary vector of $\mathcal{L}^*$. This implies, from equations (5.5,66) and (5.5,67), that there exists a matrix W such that

$$K_2 A_{22}^{i-k} = \bar{C}\,\bar{A}^{i-k} W \tag{5.5,68}$$

for $i \geq k$. It follows that

$$Q_1 = \begin{bmatrix} K_2 \\ K_2 A_{22} \\ \cdot \\ \cdot \\ \cdot \\ K_2 A_{22}^{j-k} \end{bmatrix} = \begin{bmatrix} \bar{C} \\ \bar{C}\,\bar{A} \\ \cdot \\ \cdot \\ \cdot \\ \bar{C}\,\bar{A}^{j-k} \end{bmatrix} W = Q_2 W \tag{5.5,69}$$

where $(j-k) \geq \max(d(A_{22}), d(\bar{A}))$. The pair (K_2, A_{22}) is observable by Lemma 11 so that $\rho[Q_1] = s$ where $d(\mathcal{L}^*) = s$. From (5.5,69), it follows that $\rho[W] = s$. As the pair $(\bar{C}, \bar{A})$ is completely observable, Q_2 has full column rank and, hence, a left inverse, Q_2^+. Thus, (5.5,69) yields

$$Q_2^+ Q_1 = W \tag{5.5,70}$$

Extending (5.5,69) one more step,

$$Q_1 A_{22} = Q_2 \bar{A} W \tag{5.5,71}$$

Multiply on the left by Q_2^+ and use (5.5,70):

$$WA_{22} = \bar{A} W \tag{5.5,72}$$

This expression shows that $\mathcal{V} = \mathcal{R}[W]$ is $\bar{A}$ - invariant and that

$$A_{22} = \bar{A} \mid \mathcal{V} \tag{5.5,73}$$

Thus, the characteristic polynomial of A_{22}, $\alpha_L(s)$ divides the characteristic polynomial of $\bar{A}$, $\bar{\alpha}(s)$. Since $(\bar{C},\bar{A})$ is the observable subsystem of $(\hat{C},\hat{A})$, it follows that $\bar{\alpha}(s)$ divides $\bar{\alpha}(s)$. Hence, $\alpha_L(s)$ divides $\hat{\alpha}(s)$ and the theorem is proven. □

This theorem provides a lower bound for the dimension of a minimal left inverse. This bound is $d(\mathcal{L}^*)$.

The next theorem gives an explicit representation to a minimal left inverse.

<u>Theorem 23</u>

Suppose that a system (A,B,C,D) with zero initial conditions has a left inverse. Then a minimal order left inverse of order $d(\mathcal{L}^*)$ is given by

$$\tilde{x}_{i+1} = A_{22}\tilde{x}_i + A_{21}N(p)\hat{y}_i + [B_{21} \quad B_{22}]\bar{N}(p)y_i$$

$$u_i = K_2\tilde{x}_i + K_1N(p)\hat{y}_i + \overline{N}(p)y_i$$

where the notation is taken from (5.5,55), the polynomial matrices, N(s) and $\overline{N}(s)$, are defined by Lemma 10, and p is the unit delay.

Proof: It follows from Theorem 22 that $d(\mathcal{L}^*)$ is the minimum order of any left inverse. For an input sequence define

$$v_i = u_i - Kx_i \tag{5.5,74}$$

where K is a feedback matrix such that the system can be placed in the form of (5.5,55). Place the system in the form of (5.5,55) and adopt the notation of (5.5,55). From Lemma 10, there exist matrices N(s) and $\overline{N}(s)$ such that

$$\begin{aligned} \bar{x}_i &= N(p)\hat{y}_i \\ v_i &= \overline{N}(p)y_i \end{aligned} \tag{5.5,75}$$

Substituting (5.5,75) into (5.5,55c) yields

$$\begin{aligned} u_i &= K_1\bar{x}_i + K_2\tilde{x}_i + v_i \\ &= K_2\tilde{x}_i + K_1N(p)\hat{y}_i + \overline{N}(p)y_i \end{aligned} \tag{5.5,76}$$

where p is the unit delay. From (5.5,55a), it is seen that $\tilde{x}_i$ satisfies

$$\begin{aligned} \tilde{x}_{i+1} &= A_{22}\tilde{x}_i + A_{21}\bar{x}_i + B_2v_i \\ &= A_{22}\tilde{x}_i + A_{21}N(p)\hat{y}_i + B_2\overline{N}(p)y_i \end{aligned} \tag{5.5,77}$$

$$\tilde{x}_0 = 0$$

Equations (5.5,76) and (5.5,77) are the desired result. □

Consider now the construction of a right inverses. Suppose a system (5.5,1) with zero initial conditions has a right inverse. In general, $\mathcal{R}^* \neq 0$ so that by Theorem 11, there exists a feedback matrix K such that (5.5,1) has the form

$$x_{i+1} = (A + BK)x_i + Bv_i = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \tilde{x}_i \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} \bar{v}_i \\ \tilde{v}_i \end{bmatrix} \tag{5.5,78a}$$

$$y_i = (C + DK)x_i + Dv_i = [C_1 \quad 0] \begin{bmatrix} \bar{x}_i \\ \tilde{x}_i \end{bmatrix} + [\bar{D} \quad 0] \begin{bmatrix} \bar{v}_i \\ \tilde{v}_i \end{bmatrix} \tag{5.5,78b}$$

$$u_i = Kx_i + v_i = K_1\bar{x}_i + K_2\tilde{x}_i + v_i \tag{5.5,78c}$$

(It is assumed that the necessary change of bases has been done).

<u>Lemma 12</u>

The system $\left((A + BK), \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, (C + DK), \bar{D}\right)$ has a right inverse, if the system (5.5,76) has a right inverse.

Proof: The special structure of the system in (5.5,78) permits the following calculation:

$$CA^iB = [C_1 \quad 0] \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^i \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix} = [C_1A_{11}^iB_1 \quad 0] \tag{5.5,79}$$

This shows that the block columns in the matrix $Q_{[0,i]}$ (5.5,4) have at most m_1 linearly independent columns where m_1 is the number of columns in B_1. From equation (5.5,19), it is seen that $\rho[F_i]$ is the number of linearly independent columns in that column of blocks, i.e. $\rho[F_i]$ for (5.5,78) is at most m_1. Since (5.5,78) has a right inverse, it follows from Theorem 19 that $m_1 = r$. Hence, the system $\left((A+BK), \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, (C+DK), \bar{D}\right)$ has a right inverse. □

Lemma 13

The system $\left((A+BK), \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, (C+DK), \bar{D}\right)$, obtained from (5.5,78), has a left inverse.

Proof: For the system (5.5,78),

$$\mathcal{R}^* = \langle A_{22} \mid B_4 \rangle \tag{5.5,80}$$

See equation (5.2,31). Since $\mathcal{R}^*$ is maximal for (5.5,78), it follows that $\mathcal{R}^* = 0$ for $\left((A+BK), \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, (C+DK), \bar{D}\right)$. By Corollary 3, $\left((A+BK), \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, \bar{D}\right)$ has a left inverse. □

Lemma 13 provides a method for constructing a right inverse for (5.5,78). If a system has both a left and right inverse, then these inverses are the same. By Lemma's 12 and 13, this is true for $\left((A+BK), \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, (C+DK), \bar{D}\right)$ and its inverse is given by Lemma 10.

To obtain the inverse of (5.5,78), form two matrices, S_1 and S_2, such that

$$S_1\bar{v}_i = \begin{bmatrix} I \\ \\ 0 \end{bmatrix}\bar{v}_i = \begin{bmatrix} \bar{v}_i \\ \\ 0 \end{bmatrix} \tag{5.5, 81a}$$

$$S_2\tilde{v}_i = \begin{bmatrix} 0 \\ \\ I \end{bmatrix}\tilde{v}_i = \begin{bmatrix} 0 \\ \\ \tilde{v}_i \end{bmatrix} \tag{5.5, 81b}$$

Now (5.5,78c) can be written as

$$u_i = Kx_i + v_i = K_1\bar{x}_i + K_2\tilde{x}_i + S_1\bar{v}_i + S_2\tilde{v}_i \tag{5.5, 82}$$

Substituting from Lemma 10,

$$u_i = K_1N(p)\hat{y}_i + K_2\tilde{x}_i + S_1\bar{N}(p)y_i + S_2\tilde{v}_i \tag{5.5, 83}$$

where (from 5.5, 78) $\tilde{x}_i$ satisfies

$$\begin{aligned} \tilde{x}_{i+1} &= A_{22}\tilde{x}_i + A_{21}\bar{x}_i + B_3\bar{v}_i + B_4\tilde{v}_i \\ &= A_{22}\tilde{x}_i + A_{21}N(p)\hat{y}_i + B_3\bar{N}(p)y_i + B_4\tilde{v}_i \end{aligned} \tag{5.5, 84}$$

again by Lemma 10. Equations (5.5,83) and (5.5,84) constitute a right inverse for (5.5,78). These results are summarized in the next theorem.

Theorem 24

Suppose that a system (5.5,1) with zero initial condition has a right inverse. Then a right inverse is

$$\tilde{x}_{i+1} = A_{22}\tilde{x}_i + A_{21}N(p)\hat{y}_i + B_3\bar{N}(p)y_i + B_4\tilde{v}_i$$

$$u_i = K_2\tilde{x}_i + K_1N(p)\hat{y}_i + S_1\overline{N}(p)y_i + S_2\tilde{v}_i$$

where the matrices $N(s)$ and $\overline{N}(s)$ are given by Lemma 10, p is the unit delay, and $\tilde{v}_i$ is an arbitrary input sequence. □

The construction of the left and right inverses of a system (5.5,1) as in Theorem's 23 and 24, clarifies the role of invariant zeros in inverse systems. Consider left inverses. The poles of the minimal left inverse are the eigenvalues of A_{22}. Recall that A_{22} was obtained from canonic form (5.2,24) and that $\mathcal{R}^* = 0$ for a left invertible system. By Theorem 14, the invariant zeros are just the eigenvalues of A_{22}, i.e. the invariant zeros of (5.5,1) are the poles of its minimal left inverse, if it exists. By Theorem 22, the invariant zeros are contained among the poles of any other left inverse.

Now suppose that the system (5.5,1) has a right inverse and it is given by Theorem 24. As noted in Section 5.3, the invariant zeros are the uncontrollable modes of (A_{22}, B_4). Again, the invariant zeros are found among the poles of the inverse system. This is a generalization of the well known classical result. These results are summarized in the following theorem.

Theorem 25

Consider the system (5.5,1) with zero initial conditions.

a) If (5.5,1) has a left inverse, then the invariant zeros are contained among the poles of the left inverse system. If this system has minimal order then the invariant zeros are exactly the poles of the left inverse.

b) If (5.5,1) has a right inverse system and is constructed according to Theorem 24, then the invariant zeros appear among the poles of the right inverse system. □

These results were noted by Bengtsson [2] for left inverses of systems with $D = 0$.

Actually, a slightly stronger statement can be made for systems with a left inverse. Theorem 14 states that the invariant polynomials of the system matrix for (5.5,1) are the same as the invariant polynomials of A_{22}. In Section 4.2, it was noted that the invariant zeros determine whether a system has simple or non-simple structure. This structure is determined by the invariant polynomials, and this structure carries over into the pole configuration of the left inverse system.

Of course, the stability of the inverse system is also governed by the location of the invariant zeros. For a right inverse, if $\mathcal{R}^* \neq 0$ some of the poles of the right inverse system are not invariant zeros. It is exactly those poles which are controllable from the arbitrary input. Hence, if the instability of a right inverse is not due to a right half plane invariant zero, the system can be stabilized by using the extra freedom in the control. These stability questions of inverse systems are also addressed in [40], although the role of invariant zeros is not explicit. This reference also provides a method for constructing a reduced order inverse system based on the work in [34].

Finally, it is noted that in [41] it is shown for a class of systems that the invariant zeros of the inverse system are actually the poles of the original system; an interesting generalization of a classical result.

CHAPTER 6

COMPUTATION OF ZEROS

6.1. Introduction

Several algorithms have appeared in the literature for calculating zeros ([22],[31],[32],[42],[43],[44]) based on the properties of zeros described in Chapter 5. These algorithms provide efficient methods for calculating zeros or they are readily adaptable to a digital computer. All of these algorithms utilize state space representations of systems and they all calculate invariant zeros. However, there are certain limitations and distinctions in each.

Section 2 discusses the generalized eigenvalue method. This algorithm requires that the system be nondegenerate. It can also be used to calculate decoupling zeros.

Section 3 presents two methods based on high gain feedback. The first is a straightforward application of Theorem 17. It requires that the system be nondegenerate. The second algorithm applies the idea of high gain feedback to obtain a geometric method of calculating invariant zeros.

The Davison-Wang method is introduced in Section 4. The invariant zeros are observed to be the limiting positions of the eigenvalues of the matrix

$$\begin{bmatrix} A & \gamma B \\ C & \gamma D \end{bmatrix} \tag{6.1,1}$$

where γ is a scalar that approaches infinity. The system is required to be nondegenerate.

Section 5 discusses a certain class of systems for which a right inverse system is easily constructed. For these systems, the results of Section 5.5 on inverse systems are considerably simplified. The invariant zeros are then calculated as the uncontrollable modes of the right inverse system.

6.2. Generalized Eigenvalue Approach

As a motivation for this algorithm, consider a nondegenerate state space system with an equal number of inputs and outputs. The invariant zeros are given by

$$\det \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = 0. \qquad (6.2,1)$$

Because P(s) loses rank when (6.2.1) is satisfied, it follows that

$$s \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \overline{x} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \overline{x} \qquad (6.2,2)$$

or

$$s \overline{A}\overline{x} = \overline{B}\overline{x} . \qquad (6.2,3)$$

This last equation defines a generalized eigenvalue problem. The numerical solutions to this problem have been studied and the results can be applied here to obtain the invariant zeros of a system defined by the system matrix in (6.2,1). Patel [42] first suggested this approach and proposed a numerical algorithm based on [45]. Since then it has been suggested that the QZ-algorithm be used to solve (6.2,3) [43]. The numerical advantages of the QZ-algorithm are emphasized in this reference.

This analysis has assumed that the system matrix in (6.2,1) has normal rank n+m as this is a requirement for the numerical algorithm given in [45]. If this algorithm is used, this requirement must be checked beforehand. Alternatively, if the QZ-algorithm is applied to (6.2,3), it will automatically detect degeneracy. In either case, this approach will not identify invariant zeros (Definition 5) of a degenerate system. However, it is naturally suited to calculate zeros defined by Definition 6. See comments in Section 4.2.

In [43], this method is extended to systems with an unequal number of inputs and outputs by "squaring up" the system. Suppose that $m > r$. Generate two pseudo-random matrices E_1 and F_1 such that

$$\bar{A} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} A & B \\ -C & -D \\ -E_1 & -F_1 \end{bmatrix} \tag{6.2,4}$$

where $\bar{B}$ is a $(n+m) \times (n+m)$ matrix. It can be seen, via Theorem 4, that the invariant zeros of the original system are contained in the invariant zeros of the new system. To find the invariant zeros of the original system, first calculate the invariant zeros of (6.2,4). Then generate two new matrices, E_2 and F_2, and calculate the invariant zeros again, replacing E_1 and F_1 by E_2 and F_2, respectively. Then the invariant zeros of the original system are almost surely contained in the intersection of these two sets of invariant zeros. If $r > m$, this problem can be reformulated in an analogous way, or the dual system can be considered.

The decoupling zeros can be calculated using this generalized eigenvalue technique, too. Recall that the input decoupling zeros are

those complex numbers s such that

$$s[I \vdots 0] - [A \vdots B] \tag{6.2,5}$$

loses rank (Theorem 7). Augmenting these matrices by pseudo-random matrices E_1 and F_1 of the appropriate size gives:

$$s\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ E_1 & F_1 \end{bmatrix}. \tag{6.2,6}$$

This again can be used as a generalized eigenvalue problem and solved as in the case of invariant zeros with unequal number of inputs and outputs. The output decoupling zeros are handled in a similar fashion.

Of course, once the invariant zeros and input and output decoupling zeros are known, the transmission zeros and the system zeros can be recovered using the relationships in Theorem 9.

More recently, this problem has been cast in a more general theoretical framework [9]. It has been noted that

$$P(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} \tag{6.2,7}$$

is a singular pencil of matrices [11] and that its finite divisors are just the invariant zeros. The solutions to (6.2,2) are just the roots of these finite divisors. However, in general, the number of solutions to (6.2,3) is less than the dimension of $\bar{A}$. Therefore, it is proposed in [46] that a sequence of transformations be applied to (6.2,7) that will reduce P(s) to a block canonic form. One of the blocks will contain exactly the invariant zeros (finite divisors) of (6.2,7). Then the QZ-algorithm can be applied with greater efficiency. It turns out that the

necessary transformations are exactly the structure algorithm [34] as modified in [40]. Furthermore, these transformations will reduce a nonsquare P(s) to a square matrix $\bar{P}(s)$ so that the QZ-algorithm can be applied directly. The details are found in [46].

6.3. High Gain Feedback

Several algorithms for computing invariant zeros have appeared that are based on the idea that zeros are the limiting positions of the systems poles under high gain feedback ([32],[31],[47],[48]).

The first algorithm is obtained directly from Theorem 17. It requires that the system be nondegenerate so that the algorithm first checks this condition. If the system is nondegenerate, then the invariant zeros are contained among the finite eigenvalues of

$$A + B(\frac{1}{\rho} I_r - KD)^{-1} KC \tag{6.3,1}$$

as $\rho \to \infty$. Here, K is an arbitrary $m \times r$ matrix with rank min(m,r). If $m = r$, K can be chosen to be the identity matrix. Then the invariant zeros coincide with the finite eigenvalues of (6.3,1). If $m \neq r$, then for "almost all" K the invariant zeros will be contained in the finite eigenvalues of (6.3,1). Hence, choose a K and a suitably large ρ and calculate the eigenvalues of (6.3,1). If $m \neq r$, the finite eigenvalues must also satisfy the definition of an invariant zero.

Again, the nondegeneracy condition implies that for systems for which this algorithm is applicable, Definitions 5 and 6 define the same set of zeros. It will not identify invariant zeros (Definition 5)

for degenerate systems. Hence, it is more suited for use with Definition 6. See [32] for details.

The second algorithm, however, does calculate invariant zeros (Definition 5) for degenerate systems. Suppose that $D = 0$ so that (6.3,1) reduces to

$$A + \rho BKC. \tag{6.3,2}$$

If $m = r$ (number of inputs equals the number of outputs), then it is shown in [31] that the invariant zeros tend to the roots of the equation

$$\det[sNM - NAM] = 0 \tag{6.3,3}$$

where N and M are the full-rank left and right annihilators of B and C, respectively, such that

$$\begin{aligned} NB &= 0 \\ CM &= 0 \end{aligned} \tag{6.3,4}$$

when $\rho \to \infty$. Furthermore, if the product CB has full rank, it is shown [31] that N and M can always be selected so that

$$NM = I. \tag{6.3,5}$$

In this case, the invariant zeros are just the eigenvalues of NAM. If CB does not have full rank or $m \neq r$ then suitable modifications are made in the algorithm. The details are in [31].

In [47], the following interpretation is given to the high gain feedback system (6.3,2). Suppose that $m = r$ and choose K to be the identity matrix. Then as $\rho \to \infty$ some of the eigenvalues of (6.3,2) go to infinity while others tend to finite values (the invariant zeros). Hence, there is a natural separation of the poles of the closed-loop system into

slow and fast modes as ρ increases. This heuristic argument shows that this system is naturally suited for singular perturbation analysis which is done in [47] and [49]. The results are essentially the same as for the NAM algorithm, however, this analysis does produce a specific procedure to calculate the matrices N and M when CB has full rank. See [47] or [49] for details.

Another specific construction of the matrices N and M is given in [48] for systems with equal numbers of inputs and outputs. There the NAM algorithm is extended to systems with $D \neq 0$. This provides a method for calculating system zeros as well as invariant zeros. Recall that the set of system zeros is the intersection of the sets of invariant zeros obtained from certain subsystems of the original system. (See the remark in Section 4.4 following Example 14.) So simply apply the algorithm in [48] (or any other algorithm for computing invariant zeros) to all subsystems of the form (4.4,11). The system zeros are then the invariant zeros common to all these subsystems.

6.4. The Davison-Wang Method

The following method was proposed by Davison and Wang [22]. Consider a system with an equal number of inputs and outputs which is nondegenerate. Define the matrix $S(\gamma)$ as

$$S(\gamma) = \begin{bmatrix} A & \gamma B \\ C & \gamma D \end{bmatrix}. \tag{6.4,1}$$

It is shown that as γ becomes arbitrarily large, p eigenvalues of $S(\gamma)$ become arbitrarily close to the invariant zeros of the original system.

To apply this algorithm, first determine if the system is degenerate. If it is not, then choose a suitably large γ and calculate the eigenvalues of $S(\gamma)$. The invariant zeros are those eigenvalues of $S(\gamma)$ which satisfy the definition of invariant zeros. Under the assumption of nondegeneracy, this could be either Definition 5 or 6. However, as with the generalized eigenvalue approach, this algorithm is naturally suited to Definition 6.

If the system has an unequal number of inputs and outputs, then $S(\gamma)$ can be augmented by pseudo-random matrices E_1 and γF_1 to make $S(\gamma)$ square. The procedure is exactly the same as discussed for the generalized eigenvalue problem, Section 6.2.

6.5. Calculating Invariant Zeros from Inverse Systems

The final method for computing invariant zeros is found in [44]. It is based on the results of Section 5.5 in which invariant zeros are related to the poles of the inverse system. Specifically, suppose that a left or right inverse system, as given in Theorems 23 or 24, can be constructed. Then the invariant zeros could be computed from the dynamics of the system. Of course, this construction can always be carried out as suggested by the proofs of the theorems. However, in some cases this construction is accomplished with much less effort.

First consider a system

$$x_{i+1} = Ax_i + Bu_i \tag{6.5, 1a}$$

$$y_i = Cx_i + Du_i \tag{6.5, 1b}$$

in which $m \geq r$ and D has full rank. From Algorithm I (Section 5.2), it is seen that $\rho[F_1] = r$. Therefore, by Theorem 19, (6.5,1) has a right inverse.

If, in addition, $r = m$ then $\rho[F_1] = m$ and (6.5,1) also has a left inverse. Using the notation of Lemma 10 and Theorem 23, the fact that $\rho[D] = r$ implies

$$\bar{y}_i = y_i \qquad (6.5,2)$$
$$\hat{y}_i = 0.$$

Now by Algorithm I, if $\rho[F_1] = r$, it follows that $M_1 = 0$. Hence,

$$\mathcal{N}[M_1] = \mathcal{N}[M_0] = \mathcal{X} \qquad (6.5,3)$$

where $\mathcal{X}$ is the state space. By Lemma 1 and Theorem 12

$$\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}^* = \mathcal{X}. \qquad (6.5,4)$$

Therefore, to find K, the feedback matrix which puts (6.5,1) into canonic form (5.2,24), it is necessary to solve

$$C + DK = 0. \qquad (6.5,5)$$

First assume $m = r$. Then D^{-1} exists and (6.5,5) yields

$$K = -D^{-1}C. \qquad (6.5,6)$$

From canonic form (5.2,24)

$$A_{22} = A + BK = A - BD^{-1}C. \qquad (6.5,7)$$

Under the assumptions above, from Lemma 10 it follows that

$$\bar{N}(p) = D^{-1} \qquad (6.5,8)$$
$$N(p) = 0.$$

Now, Theorem 23 gives the left inverse for (6.6,1), assuming that $m = r$ and D has full rank,

$$x_{i+1} = (A-BD^{-1}C)x_i + BD^{-1}y_i$$
$$u_i = -D^{-1}Cx_i + D^{-1}y_i. \qquad (6.5,9)$$

Hence, the invariant zeros are the eigenvalues of $(A-BD^{-1}C)$ which is one of the results given in [44].

Now suppose that $m > r$ and D has full rank. Equations (6.5,2)-(6.5,4) still hold and the required matrix K must still solve (6.5,5). The procedure given in [44] is this: Write (6.5,5) as

$$Cx_i + Du_i = y_i. \qquad (6.5,10)$$

A general solution to this equation is

$$u_i = [(I_m-D^+D)L - D^+C]x_i + D^+y_i \qquad (6.5,11)$$

where L is an arbitrary $m \times n$ matrix to provide degrees of freedom and

$$D^+ = D^+(DD^+)^{-1}. \qquad (6.5,12)$$

Substitute for u_i in (6.5,1a)

$$x_{i+1} = \{(A-BD^+C) + B(I_m-D^+D)L\}x_i + BD^+y_i. \qquad (6.5,13)$$

To find the invariant zeros, calculate the eigenvalues of

$$(A-BD^+C) + B(I_m-D^+D)L \qquad (6.5,14)$$

for two values of L, say zero and L "larger." The invariant zeros are the eigenvalues of (6.5,14) which are the same for both values of L.

To see how this fits into the theory of inverse systems, perform an input space transformation to bring (6.5,1) into the form

$$x_{i+1} = Ax_i + [B_1 \quad B_2]\begin{bmatrix}\bar{u}_i \\ \tilde{u}_i\end{bmatrix}$$
$$y_i = Cx_i + [\bar{D} \quad 0]\begin{bmatrix}\bar{u}_i \\ \tilde{u}_i\end{bmatrix} \qquad (6.5,15)$$

where $\bar{D}$ is a $r \times r$ nonsingular matrix and B is partitioned compatibly. A straightforward calculation shows that (6.5,12) yields

$$D^+ = \begin{bmatrix} \bar{D}^{-1} \\ 0 \end{bmatrix}. \qquad (6.5,16)$$

In particular, $K = -D^+$, where D^+ is given in (6.5,16), solves (6.6,6). Lemma 10 again yields

$$\bar{N}(p) = \bar{D}^{-1} \qquad (6.5,17)$$
$$N(p) = 0.$$

Now Theorem 24 gives a right inverse for (6.6,15)

$$x_{i+1} = (A-BD^TC)x_i + B_1\bar{D}^{-1}y_i + B_2\tilde{u}_i. \qquad (6.5,18)$$

Again, a straightforward calculation using (6.5,15) and (6.5,16) shows

$$B(I_m-D^+D)L = [B_1 \;\; B_2] \left[I_m - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \right] L$$

$$= [0 \;\; B_2] L. \qquad (6.5,19)$$

So, L can be interpreted as a feedback matrix between $(A-BD^TC)$ and B_2 in (6.5,18). Since $\mathcal{L}^* = \mathcal{X}$ for (6.5,15), it follows from (5.2,31) that

$$\mathcal{R}^* = \langle (A-BD^+C) | \mathcal{R}[B_2] \rangle. \qquad (6.5,20)$$

Then the invariant zeros of (6.5,15) are the uncontrollable modes of $((A-BD^+C),B_2)$ by Theorem 14. By choosing those eigenvalues of $((A-BD^+C)+B(I_m-D^+D)L)$ which are invariant to choices of L, the eigenvalues of the uncontrollable modes of $((A-BD^+C),B_2)$ are selected. These are exactly the invariant zeros of (6.5,15).

Finally, suppose $r > m$ for (6.5,1) and D has full rank. Then, the invariant zeros can be calculated by applying the procedure of (6.5,10)-(6.5,14) to the dual system (A^T, C^T, B^T, D^T).

The method given in [44] can be applied to systems (6.5,1) for which $D \equiv 0$. It is required that B and C have full rank and that CB have full rank.

Define a similarity transformation

$$z_i = Tx_i \tag{6.5,21}$$

where

$$T = \begin{bmatrix} C_1 & C_2 \\ 0 & I_{n-r} \end{bmatrix} \tag{6.5,22}$$

$$C = [C_1 \quad C_2].$$

Since T must be nonsingular, C_1 must be nonsingular. As C has full rank, this can always be accomplished by permuting the state variables if necessary. The application of (6.5,21) to (6.5,1) yields

$$z_{i+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z_i + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i \tag{6.5,23}$$

$$y_i = [I_r \quad 0] z_i.$$

In [50], it is shown that the transfer function of (6.5,23) can be written as

$$G(s) = Q^{-1}(s)W(s) \tag{6.5,24}$$

where

$$Q(s) = sI - A_{11} - A_{12}(sI-A_{22})^{-1}A_{21} \tag{6.5,25a}$$

$$W(s) = B_1 + A_{12}(sI-A_{22})^{-1}B_2. \tag{6.5,25b}$$

The system matrix of (6.5,23) is given by

$$P(s) = \begin{bmatrix} sI-A_{11} & -A_{12} & -B_1 \\ -A_{21} & sI-A_{22} & -B_2 \\ I_r & 0 & 0 \end{bmatrix}. \tag{6.5,26}$$

Using elementary row and column operations it can be brought into the form

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & sI-A_{22} & -B_2 \\ 0 & -A_{12} & B_1 \end{bmatrix}. \tag{6.5,27}$$

By Definition 5, the invariant zeros of (6.5,23) are the roots of the invariant polynomials of the matrix (6.5,27). On the other hand, the lower right hand corner of (6.5,27) can be interpreted as the system matrix of the system (A_{22},B_2,A_{12},B_1), whose transfer function is given in (6.5,25b). Hence, the invariant zeros of (6.5,23) are the same as the invariant zeros of the reduced order system (A_{22},B_2,A_{12},B_1) and the transmission zeros of (6.5,23) are the same as the reduced order system whose transfer function matrix is given by (6.5,25b) (Theorem 6).

To actually calculate the invariant zeros of (A_{22},B_2,A_{12},B_1) note that the transformation in (6.2,22) yields

$$TB = \begin{bmatrix} CB \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{6.5,28}$$

Since CB was assumed to have full rank, B_1 has full rank. Therefore, the invariant zeros of (A_{22},B_2, A_{12},B_1) can be calculated using (6.5,7), the procedure given in (6.5,10)-(6.5,14), or this procedure applied to the dual system $(A_{22}^T,A_{12}^T,B_2^T,B_1^T)$.

More recently, this procedure for the calculation of invariant zeros has been extended to the class of all invertible systems. This was accomplished by an algorithm which is a variation of Silverman's Structure Algorithm [34]; i.e. by augmenting the D matrix to make it have full rank. The details are in [51].

CHAPTER 7

A NEW ALGORITHM FOR CALCULATING INVARIANT ZEROS

7.1. Introduction

This chapter presents an algorithm for calculating invariant zeros. The algorithm is based on the geometrical properties of linear time-invariant systems; specifically, it calculates canonic form (5.2.24) which displays explicitly $\mathcal{L}^*$, the maximal null-output (A,B)-invariant subspace. Once this canonic form is available, the invariant zeros are easily calculated via Theorem 14.

The algorithm is developed using the geometrical techniques introduced in Section 5.2. The algorithm proceeds by identifying the subspaces $\mathcal{L}_i$ for $i = 0,1,2,\ldots$. This is accomplished by applying a sequence of similarity transformations and feedback matrices to the system (A,B,C,D). The algorithm terminates when $\mathcal{L}_i = \mathcal{L}_{i+1} = \mathcal{L}^*$ and the system is in canonic form (5.2.24). This form then can be further reduced to canonic form (5.2.38).

Section 2 developes the algorithm, called Algorithm II, for calculating canonic form (5.2.24). Algorithm II is closely related to Wonham's vector space algorithm for calculating $\mathcal{L}^*$ [12], Silverman's structure a algorithm [34], and Algorithm I (Section 5.2). This relationship is discussed in detail. Other methods for calculating $\mathcal{L}^*$ are also briefly discussed.

Section 3 explains how the canonic form produced can be used to calculate invariant zeros. Since Algorithm II can be used on any system

(A,B,C,D), the invariant zeros can be calculated for any system (A,B,C,D). This method of computing invariant zeros is compared to other methods which were discussed in Chapter 6. Algorithm II is then summarized and several examples are given to illustrate the use of Algorithm II.

7.2. An Algorithm for Calculating $\mathcal{L}^*$

Theorem 11 states that any system (A,B,C,D) can be placed in this canonic form:

$$\begin{bmatrix} \bar{x}_{i+1} \\ \tilde{x}_{i+1} \\ \hline y_i \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ A_3 & A_4 & B_3 & B_4 \\ \hline C_1 & 0 & D_1 & 0 \end{array}\right] \begin{bmatrix} \bar{x}_i \\ \tilde{x}_i \\ \bar{u}_i \\ \tilde{u}_i \end{bmatrix} . \qquad (7.2.1)$$

However, this theorem does not provide the required state feedback matrix, or the input and state space transformation matrices. The purpose of the following algorithm is to provide a method for placing a completely arbitrary system (A,B,C,D) in canonic form (7.2.1).

The algorithm proceeds in cycles; the i-th cycle identifying the subspace $\mathcal{L}_i$ (Definition 11). By Lemma 1 when $\mathcal{L}_i = \mathcal{L}_{i+1}$, $\mathcal{L}_i = \mathcal{L}^*$, and the system will be in canonic form (7.2.1). Recall that Algorithm I also calculated $\mathcal{L}^*$. This algorithm is based on its properties.

Algorithm II

1st Cycle

Consider the system (A,B,C,D). Since it is desired to calculate $\mathcal{L}_1$, apply Steps 1 and 2 of Algorithm I to (A,B,C,D). From (5.2.15)

$$S_o\Gamma_o = S_o \begin{bmatrix} 0 & 0 \\ D & C \end{bmatrix} = \begin{bmatrix} F_1 & G_1 \\ 0 & M_1 \end{bmatrix}. \tag{7.2.2}$$

The idea is to express this calculation in terms of a similarity transformation on the original system. To this end, find a nonsingular input space transformation W_o such that

$$DW_o = [(D^1)' \; 0] = D^1 \tag{7.2.3}$$*

where $(D^1)'$ is a $r\times m_1$ matrix which has full column rank. Denote $B^1 = BW_o$ and $u_i^1 = W_o u_i$. This operation corresponds to postmultiplication of (7.2.2) by

$$\begin{bmatrix} W_o & 0 \\ 0 & I_n \end{bmatrix}. \tag{7.2.4}$$

Since W_1 is nonsingular, the row rank of F_1 is not affected and the algorithm is unchanged.

Now define a nonsingular output space transformation S_o such that

$$S_o(D^1)' = \begin{bmatrix} \bar{D}^1 \\ 0 \end{bmatrix} \tag{7.2.5}$$

where $\bar{D}^1$ is a $m_1 \times m_1$ nonsingular matrix. Then

$$\begin{aligned} y_i^1 = S_o y_i = \begin{bmatrix} \bar{y}_i^1 \\ \tilde{y}_i^1 \end{bmatrix} &= S_o C x_i + S_o D^1 u_i^1 \\ = C^1 x_i + D^2 u_i^1 &= \begin{bmatrix} \bar{C}^1 \\ \tilde{C}^1 \end{bmatrix} x_i + \begin{bmatrix} \bar{D}^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_i^1 \\ \tilde{u}_i \end{bmatrix} \end{aligned} \tag{7.2.6}$$

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*In this section, superscripts indicate indices, not exponents.

where $\bar{C}^1$ is a $m_1 \times n$ matrix, $\tilde{C}^1$ is a $r_1 \times n$ matrix, and $r_1 = r - m_1$. Furthermore, y_i^1 and u_i^1 are partitioned compatibly. Clearly, the matrix S_o defined in (7.2.4) could be used in (7.2.2). In that case, by comparing (7.2.2) and (7.2.5), it follows that $M_1 = \tilde{C}^1$.

<u>Remark 1</u>: It will be required below that $\tilde{C}^1$ have full row rank. If this is not true, then there exists a nonsingular matrix E_o' such that

$$E_o'\tilde{C}^1 = \begin{bmatrix} (\tilde{C}^1)' \\ 0 \end{bmatrix} \tag{7.2.7}$$

where $(\tilde{C}^1)'$ has full row rank and E_o' is a $r_1 \times r_1$ matrix. Let

$$E_o = \begin{bmatrix} I_{m_1} & 0 \\ 0 & E_o' \end{bmatrix} \tag{7.2.8}$$

and define $S_o' = E_o S_o$. Clearly, D^2 is not affected when S_o is replaced by S_o'. Now it is seen that $M_1 = (\tilde{C}^1)'$. Notice that

$$\eta[M_1] = \eta[\tilde{C}^1] = \eta[(\tilde{C}^1)']. \tag{7.2.9}$$

Assume now that $\tilde{C}^1$ has full rank with the understanding that if this is not true, the necessary modifications can be made as per this remark. □

To obtain canonic form (7.2.1), it is necessary to find K^*; that is, a feedback matrix that will make $\mathcal{L}^*$ $(A+BK^*)$-invariant. This is accomplished by finding a matrix K_i in each cycle and calculating $A+BK_i$. When the algorithm terminates, K^* can be recovered, if necessary. To this end let

$$\bar{K}_o = -(\bar{D}^1)^{-1}\bar{C}^1. \tag{7.2.10}$$

Then form the $m \times n$ matrix K_o as

$$K_o = \begin{bmatrix} \bar{K}_o \\ 0 \end{bmatrix}. \tag{7.2.11}$$

Compute

$$A^1 = A + B^1 K_o \tag{7.2.12}$$

$$\begin{bmatrix} \bar{C}^2 \\ \tilde{C}^2 \end{bmatrix} = C^2 = C^1 + D^2 K_o = \begin{bmatrix} \bar{C}^1 \\ \tilde{C}^1 \end{bmatrix} + \begin{bmatrix} \bar{D}^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{K}_o \\ 0 \end{bmatrix}.$$

By Lemma 2, this leaves $\mathcal{L}^*$ unchanged. (Note that $\tilde{C}^2 = \tilde{C}^1$.)

Now the system is in a form such that $\mathcal{L}_1$ can be displayed explicitly. If $\tilde{C}^1 = 0$, or does not exist (D has full row rank), then from (7.2.10) and (7.2.12), it follows that $\tilde{C}^2 = 0$. Since

$$\mathcal{N}[M_1] = \mathcal{N}[\tilde{C}^1] = \mathcal{N}[\tilde{C}^2] \tag{7.2.13}$$

this implies that

$$\mathcal{L}_1 = \mathcal{N}[M_1] = \mathcal{X} \tag{7.2.14}$$

where $\mathcal{X}$ is the whole state space. Furthermore, as $\mathcal{L}_o = \mathcal{N}[M_o] = \mathcal{N}[M_1] = \mathcal{L}_1$, by Lemma 1 it is seen that

$$\mathcal{L}_o = \mathcal{X} = \mathcal{L}^*. \tag{7.2.15}$$

Hence, the system (A^1, B^1, C^2, D^2) given by

$$\begin{aligned} A^1 &= [A^1] & B^1 &= [B^1_{11} \quad B^1_{12}] \\ C^2 &= [0] & D^2 &= [D^2_{11} \quad 0\,] \end{aligned} \tag{7.2.16}$$

is in canonic form (7.2.1) with $K_o = K^*$. Therefore, the algorithm terminates.

If $\tilde{C}^1 \neq 0$, then $\mathcal{L}_o \neq \mathcal{L}_1$. To display $\mathcal{L}_1$ explicitly, define the similarity transformation

$$z^1_i = T_o x_i \tag{7.2.17}$$

where T_o is the $n \times n$ matrix

$$T_o = \begin{bmatrix} \tilde{C}_{11}^2 & \tilde{C}_{12}^2 \\ 0 & I_{n-r_1} \end{bmatrix} \tag{7.2.18}$$

$$\tilde{C}^2 = [\tilde{C}_{11}^2 \quad \tilde{C}_{12}^2]$$

and T_o is nonsingular.

Remark 2: If T_o is to be nonsingular, $\tilde{C}_{11}^2$ must be nonsingular. By Remark 1, $\tilde{C}^2$ has full row rank. Therefore, if $\tilde{C}_{11}^2$ is singular, it can be made nonsingular by a simple permutation of the state variables. This can be expressed as a similarity transformation of the type (7.2.17), however the details are omitted for clarity of presentation. □

From (7.2.5), (7.2.17), and (7.2.18) it is seen that

$$z_i^1 = T_o x_i = \begin{bmatrix} \tilde{y}_i^1 \\ x_i^1 \end{bmatrix}. \tag{7.2.19}$$

Calculate $A^2 = T_o A^1 T_o^{-1}$, $B^2 = T_o B^1$, and $C^3 = C^2 T_o^{-1}$. The system matrices now have the form

$$\begin{bmatrix} \tilde{y}_{i+1}^1 \\ x_{i+1}^1 \\ \hline \bar{y}_i^1 \\ \tilde{y}_i^1 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{11}^2 & A_{12}^2 & B_{11}^2 & B_{12}^2 \\ A_{21}^2 & A_{22}^2 & B_{21}^1 & B_{22}^1 \\ \hline 0 & 0 & \bar{D}^1 & 0 \\ I_{r_1} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \tilde{y}_i^1 \\ x_i^1 \\ \bar{u}_i^1 \\ \tilde{u}_i^1 \end{bmatrix} \tag{7.2.20}$$

where A_{11}^2 is a $r_1 \times r_1$ matrix. Since T_o is nonsingular, it follows from (7.2.13) that

$$\eta[M_1] = \eta[\tilde{C}^2] = \eta[\tilde{C}^3] \tag{7.2.21}$$

where

$$\tilde{C}^3 = [I_{r_1} \quad 0]. \tag{7.2.22}$$

Now, it is obvious from (7.2.20) that

$$\mathcal{N}[M_1] = \mathcal{N}[\tilde{C}^3] = \mathcal{L}_1 = \begin{bmatrix} 0 \\ x_i^1 \end{bmatrix}. \tag{7.2.23}$$

This completes the 1st cycle of the algorithm. To summarize the development thus far, motivate the next step of the algorithm, and lend further insight into the system structure, consider again (7.2.20). For $i = 0$

$$\begin{aligned} \tilde{y}_o^1 &= A_{11}^2 \tilde{y}_o^1 + A_{12}^2 x_o^1 + B_{11}^2 \bar{u}_o^1 + B_{12}^2 \tilde{u}_o^1 \\ x_1^1 &= A_{21}^2 \tilde{y}_o^1 + A_{22}^2 x_o^1 + B_{21}^1 \bar{u}_o^1 + B_{22}^1 \tilde{u}_o^1 \\ \bar{y}_o^1 &= \bar{D}^1 \bar{u}_o^1 \\ \tilde{y}_o^1 &= \tilde{y}_o^1. \end{aligned} \tag{7.2.24}$$

Now Definition 11 says that $\mathcal{L}_1$ is the set of initial conditions for (7.2.24) for which there exists a control u_o^1 such that $y_o^1 = 0$. The structure of the equations in (7.2.24) immediately imply $\bar{u}_o^1 = 0$ since $\bar{D}^1$ is nonsingular and

$$x_o \in \left\{ \begin{bmatrix} 0 \\ x_o^1 \end{bmatrix} \right\} = \mathcal{L}_1 \tag{7.2.25}$$

which was the result obtained in (7.2.23). Substitution of these results into (7.2.24) yields

$$\begin{aligned} \tilde{y}_1^1 &= A_{12}^2 x_o^1 + B_{12}^2 \tilde{u}_o^1 \\ x_1^1 &= A_{22}^2 x_o^1 + B_{22}^1 \tilde{u}_o^1 \\ y_o^1 &= 0. \end{aligned} \tag{7.2.26}$$

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As yet $\tilde{u}_o^1$ is unspecified. Indeed, an arbitrary choice of $\tilde{u}_o^1$ will not affect $\mathcal{L}_1$. However, note that $\tilde{y}_1^1$ will appear in the system output in the next time unit. Hence, the definition of $\mathcal{L}_2$ will require that $\tilde{y}_1^1 = 0$. The equations in (7.2.26) can be thought of as a reduced system $(A_{22}^2, B_{22}^1, A_{12}^2, B_{12}^2)$ for which it is desired to identify $\mathcal{L}_1$. This is exactly the problem which was solved by the first cycle of the algorithm. The second cycle, then, is just the analysis of the first cycle applied to the subsystem $(A_{22}^2, B_{22}^1, A_{12}^2, B_{12}^2)$. However, since the ultimate goal of this algorithm is to produce canonic form (7.2.1), it is necessary to embed the transformations of this subsystem in larger transformations which can be applied to the whole system such that the already defined structure is preserved. The necessary calculations for the next cycle are given to illustrate this procedure.

2nd Cycle

Find a $(n-m_1) \times (n-m_1)$ nonsingular matrix $\bar{W}_1$ such that

$$B_{12}^2 \bar{W}_1 = [\,(B_{12}^3)' \;\; 0\,] \tag{7.2.27}$$

where $(B_{12}^3)'$ is a $r_1 \times m_2$ matrix that has full column rank. This can be embedded in an input space transformation $u_i^2 = W_1 u_i^1$ where the $m \times m$ matrix W_1 is given by

$$W_1 = \begin{bmatrix} I_{m_1} & 0 \\ 0 & \bar{W}_1 \end{bmatrix} \tag{7.2.28}$$

Applying (7.2.28) to (7.2.20) gives

$$B^3 = B^2 W_1 = \begin{bmatrix} B_{11}^2 & (B_{12}^3)' \;\; 0 \\ B_{21}^1 & B_{22}^3 \;\; B_{23}^3 \end{bmatrix} \tag{7.2.29}$$

Note that $D^2W_1 = D^2$ as can be seen directly from (7.2,28) and (7.2,20). Next find a $m_2 \times m_2$ non-singular matrix $\bar{S}_1$ such that

$$\bar{S}_1(B^3_{12})' = \begin{bmatrix} \bar{B}^4_{12} \\ \\ 0 \end{bmatrix} \qquad (7.2,30)$$

where $\bar{B}^4_{12}$ is a non-singular $m_2 \times m_2$ matrix.

In terms of the system (7.2,20), (7.2.30) can be implemented by applying the similarity transformation $z^2_i = V_1 z^1_i$ where the $n \times n$ non-singular matrix V_1 is defined as

$$z^2_i = V_1 z^1_i = \begin{bmatrix} \bar{S}_1 & 0 \\ \\ 0 & I_{n-r_1} \end{bmatrix} \begin{bmatrix} \tilde{y}^1_i \\ \\ x^1_i \end{bmatrix} = \begin{bmatrix} \hat{y}^1_i \\ \\ x^1_i \end{bmatrix} \qquad (7.2,31)$$

Calculate $A^3 = V_1 A^2 V_1^{-1}$, $B^4 = V_1 B^3$, and $C^3 = C^2 V_1^{-1}$. The matrices A^3 and B^4 now have the form

$$A^3 = \begin{bmatrix} \bar{A}^3_{11} & \bar{A}^3_{12} \\ \tilde{A}^3_{11} & \tilde{A}^3_{12} \\ A^3_{21} & A^2_{22} \end{bmatrix}; \quad B^4 = \begin{bmatrix} \bar{B}^4_{11} & \bar{B}^4_{12} & 0 \\ \tilde{B}^4_{11} & 0 & 0 \\ B^1_{21} & B^3_{22} & B^3_{23} \end{bmatrix} \qquad (7.2.32)$$

(Here, Remark 1, which applies to $\tilde{C}^1$, also applies to $\tilde{A}^3_{12}$. If $\tilde{A}^3_{12}$ doesn't have full row rank, the necessary adjustments should be made). Define the row rank of $\tilde{A}^3_{12}$ to be r_2.

From (7.2,32), the feedback matrix K_1 can now be calculated (following (7.2,10)) as

$$\bar{K}_1 = -(\bar{B}^4_{12})^{-1}\tilde{A}^3_{12}$$

$$K_1 = \begin{bmatrix} 0_{m_1 \times r_1} & 0 \\ 0 & \bar{K}_1 \\ 0 & 0 \end{bmatrix} \tag{7.2,33}$$

where $\bar{K}_1$ is a $m_2 \times (n - r_1)$ matrix and K_1 is a $m \times n$ matrix. (Recall that A^3_{12} and B^4_{12} are the "C" and "D" matrices of the subsystem (7.2,26) to which the 2nd cycle of the algorithm is being applied). Calculate $A^4 = A^3 + B^4 K_1$ and note that $C^3 = C^3 + D^2 K_1$.

Now the matrix A^4 has the form

$$A^4 = \begin{bmatrix} \bar{A}^3_{11} & 0 \\ \tilde{A}^3_{11} & \tilde{A}^3_{12} \\ A^3_{21} & A^4_{22} \end{bmatrix} \tag{7.2,34}$$

If $\tilde{A}^3_{12} = 0$, then it follows that $\mathcal{L}_2 = \mathcal{L}_1 = \mathcal{L}^*$ by the same argument that gave (7.2,15). In this case (A^4, B^4, C^3, D^2) is in canonic form (7.2,1) and the algorithm terminates.

If $\tilde{A}^3_{12} \neq 0$, then proceed as above to identify $\mathcal{L}_2$. Define

$$T_1 = \left[\begin{array}{c|c} I_{r_1} & 0 \\ \hline 0 & \begin{array}{c} \tilde{A}^3_{12} \\ \hline \begin{array}{c|c} 0 & I \end{array} \end{array} \end{array}\right] \tag{7.2,35}$$

where T_1 is a $n \times n$ non-singular matrix. The comments in Remark 2 show that this is always possible. Form the similarity transformation

$$z_i^3 = T_1 z_i^2 = \begin{bmatrix} \hat{y}_i^1 \\ \tilde{y}_i^2 \\ x_i^2 \end{bmatrix} \qquad (7.2,36)$$

where $\tilde{y}_i^2 = \tilde{A}_{12}^3 x_i^1$. Calculate $A^5 = T_1 A^4 T_1^{-1}$, $B^5 = T_1 B^4$, and note that C^3 and D^2 are unchanged by this transformation. The system now has the form

$$\begin{bmatrix} \hat{y}_{i+1}^{1a} \\ \hat{y}_{i+1}^{1b} \\ \tilde{y}_{i+1}^2 \\ x_{i+1}^2 \\ \hline \bar{y}_i^1 \\ \tilde{y}_i^1 \end{bmatrix} = \left[\begin{array}{ccc|ccc} \bar{A}_{11}^3 & 0 & 0 & \bar{B}_{11}^4 & \bar{B}_{12}^4 & 0 \\ \tilde{A}_{11}^3 & I_{r_2} & 0 & \tilde{B}_{11}^4 & 0 & 0 \\ A_{21}^5 & A_{22}^5 & A_{23}^5 & B_{21}^5 & B_{22}^5 & B_{23}^5 \\ A_{31}^5 & A_{32}^5 & A_{33}^5 & B_{31}^2 & B_{32}^3 & B_{33}^3 \\ \hline 0 & 0 & 0 & \bar{D}^1 & 0 & 0 \\ \bar{S}_1^{-1} & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{y}_i' \\ \tilde{y}_i^2 \\ x_i^2 \\ \bar{u}_i^1 \\ \bar{u}_i^2 \\ \tilde{u}_i^2 \end{bmatrix} \qquad (7.2,37)$$

By inspection

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} 0 \\ \\ x_i^2 \end{bmatrix} \right\} \qquad (7.2,38)$$

The algorithm now enters the third cycle with the new subsystem $(A_{33}^5, B_{33}^5, A_{23}^5, B_{23}^5)$.

Clearly, this algorithm must terminate after a finite number of steps (at most n). Then the system matrices have the following form:

$$A = \left[\begin{array}{ccccc|cc} A_{11} & S_2' & 0 & \cdots & & 0 & 0 \\ A_{21} & A_{22} & S_3' & & & & \\ \vdots & & \ddots & \ddots & & & \\ \vdots & & & \ddots & S_i' & 0 & 0 \\ \vdots & & & & A_{i,i} & 0 & 0 \\ \hline A_{i+1,1} & \cdots & & \cdots & A_{i-1,i} & A_{i+1,i+1} & \end{array}\right]$$

$$B = \left[\begin{array}{ccccc|c} B_{11} & B_{12} & 0 & \cdots & & 0 \\ B_{21} & B_{22} & B_{23} & 0 & & \vdots \\ & & & & B_{i,i+1} & 0 \\ \hline B_{i+1,1} & \cdots & & \cdots & B_{i+1,i+1} & B_{i+1,i+2} \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} 0 & 0 & \cdots & 0 \\ \overline{S}_1^{-1} & 0 & \cdots & 0 \end{array}\right] \tag{7.2,39}$$

$$D = \left[\begin{array}{ccc|c} \overline{D}_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{array}\right]$$

where $A_{j,j}$ is a $r_j \times r_j$ matrix and $B_{j,j}$ is a $r_j \times m_j$ matrix. The matrices S_j' in the A matrix have the form

$$S_j' = \begin{bmatrix} 0 \\ \overline{S}_j^{-1} \end{bmatrix} \tag{7.2,40}$$

where $\overline{S}_j$ is a $r_{j+1} \times r_{j+1}$ matrix. The matrices $B_{j,j+1}$ in the B matrix have the form

$$B_{j,j+1} = \begin{bmatrix} \bar{B}_{j,j+1} \\ 0 \end{bmatrix} \qquad (7.2,41)$$

where $\bar{B}_{j,j+1}$ is a $m_{j+1} \times m_{j+1}$ matrix. This is exactly the canonic form desired. (For example, by comparing (7.2,1) to (7.2,39) it is seen that $A_4 = A_{i+1,i+1}$ and $B_4 = B_{i+1,i+2}$). □

At this point it is useful to discuss the relationship of Algorithm II to the procedure for obtaining the Generalized Hessenberg Representation (GHR) [52]. The relationship can be seen from the transformation matrices T_i, $i = 0,1,..$ defined in Algorithm II (7.2,18). Note that the matrices T_i used in Algorithm II have the same form as the transformation matrices used in obtaining the GHR of a system except that the matrices T_i are formed using only a block of the "C" matrix, not the entire "C" matrix as in obtaining the GHR. Furthermore, that portion of the "C" matrix that is used in T_i is exactly the portion which cannot be cancelled using state feedback. Hence, the GHR of the system can be obtained by using Algorithm II when it is assumed that $B = 0$ and $D = 0$. The unobservable subspace can be identified directly from the GHR of the system. On the other hand, Algorithm II identifies the largest subspace which can be made unobservable using state feedback.

Note that by duality, the transformations that give the GHR can be used to identify the unreachable subsystem of a given system. This motivates the following extension of Algorithm II which further reduces (7.2,37). By Corollary 1, canonic form (7.2,1) can be reduced to

$$\begin{bmatrix} \bar{x}_{i+1} \\ \hat{x}_{i+1} \\ \tilde{x}_{i+1} \\ y_i \end{bmatrix} = \left[\begin{array}{ccc|cc} A_1 & 0 & 0 & B_1 & 0 \\ \hat{A}_3 & A_5 & A_6 & \hat{B}_3 & B_5 \\ \tilde{A}_3 & 0 & A_7 & \tilde{B}_3 & 0 \\ \hline C_1 & 0 & 0 & D_1 & 0 \end{array}\right] \begin{bmatrix} \bar{x}_i \\ \hat{x}_i \\ \tilde{x}_i \\ \bar{u}_i \\ \tilde{u}_i \end{bmatrix} \tag{7.2,42}$$

This is just the decomposition of (A_4,B_4) in (7.2,1) into its reachable and unreachable subsystems. The following algorithm, based on the transformations to obtain the GHR, is proposed to accomplish this decomposition. The algorithm is a sequence of similarity transformations applied to the subsystem (A_4,B_4). As in Algorithm II, these similarity transformations can be embedded in a larger similarity transformation which can be applied to the full system (7.2,42). For notational simplicity this will be left to the reader.

<u>Algorithm II</u> (continued)

Let $A = A_4$ and $B = B_4$ and consider

$$w_{i+1} = Aw_i + Bw_i \tag{7.2,43}$$

Define a transformation R_1 such that $w_i = R_1 w_i^1$ where

$$R_1 = \begin{bmatrix} B_1 & 0 \\ B_2 & I \end{bmatrix} \tag{7.2,44}$$

where R_1 is a square non-singular matrix. This implies that B_1 must be non-singular. If B has full column rank, this can be achieved by a permutation of the state variables. If B doesn't have fullrank, there exist linearly dependent controls with respect to the full B and D matrices in

(7.2,42). There is no loss of generality to exclude this case. Hence, a non-singular R_1 always exists.

Now consider

$$w_{i+1} = A^T w_i, \; y_i = B^T w_i \qquad (7.2,45)$$

The first step in finding the GHR for (7.2,45) is the application of the similarity transformation $w_i^1 = R_1{}^T w_i$. The procedure for obtaining the GHR of (7.2,45) will generate a sequence of transformations R_i^T and transform (7.2,45) to explicitly display its unobservable subsystem. By duality, the transformations $w_i^{j-1} = R_j w_i^j$ will transform (7.2,43) so as to explicitly display its unreachable subsystem. The system (7.2,43) after transformation will have the form

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1,i} \\ A_{21} & A_{22} & & & \vdots \\ 0 & A_{32} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & A_{i,i-1} & A_{i,i} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (7.2,46)$$

If $A_{i,i-1} = 0$, then $A_{i,i}$ represents an unreachable subsystem. If $A_{i,i-1} \neq 0$, then the pair $(\bar{A},\bar{B})$ is reachable. In any case, (7.2,46) combined with (7.2,39) gives canonic form (7.2,42). (For example, if $A_{i,i-1} = 0$, then $A_{i,i} = A_7$). □

Algorithm II is closely related to three other algorithms which have appeared in the literature: Algorithm I [26] (Section 5.2), Silverman's Structure Algorithm [34], and Wonham's algorithm for calculating $\mathcal{L}^*$[12].

The relationship between Algorithm I and Algorithm II was detailed in the development of the first cycle of Algorithm II and will not be discussed further.

Using the notation of Algorithm I, the Structure Algorithm can be described as follows:

<u>Structure Algorithm</u>:

Step 0: Set $i=0$, $\Delta_0=0$, $F_0=D$, and $G_0=C$

Step 1: Determine any nonsingular S_i such that

$$S_i \begin{bmatrix} \Delta_i B & \Delta_i A \\ F_i & G_i \end{bmatrix} = \begin{bmatrix} F_{i+1} & G_{i+1} \\ 0 & \Delta_{i+1} \end{bmatrix}$$

where F_{i+1} has full row rank.

Step 2: Set $i=i+1$ and go to Step 1. □

In [26] the following relationship between Algorithm I and the Structure Algorithm is proved:

$$M_i = \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_i \end{bmatrix} \tag{7.2,47}$$

This implicitly establishes the connection between Algorithm II and the Structure Algorithm. Actually a closer connection can be established but the details will not be given here. Suffice it to notice that $\Delta_1 = \tilde{C}^1$ (7.2,6) and that Δ_2 is related to

$$[\tilde{A}^3_{11} \quad \tilde{A}^3_{12}] \tag{7.2,48}$$

(7.2,34) through similarity transformations. Also note that when the Structure Algorithm is applied to a system (A,B,C,D), at the i-th step the system has the form

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ y_i &= G_i x_i + F_i u_i \end{aligned} \tag{7.2,49}$$

Hence, (7.2,49) is not equivalent to the original system in the sence that (7.2,49) can't be obtained from (A,B,C,D) by change of basis in input, output, and state spaces or by the application of state feedback. In this respect, Algorithm II and the Structure Algorithm differ.

Algorithm I and Algorithm II above generate a sequence of subspaces $\mathcal{L}_i$. It can be shown [26] that the subspaces $\mathcal{L}_i$ satisfy the following vector space algorithm:

$$\mathcal{V}^{i+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}^i + \mathcal{R} \begin{bmatrix} B \\ D \end{bmatrix} \right\} \tag{7.2,50}$$

$$\mathcal{V}^0 = \mathcal{X}$$

(The proof is similar to the proof of the following theorem). This algorithm is a generalization of Wonham's algorithm for the calculation of $\mathcal{L}^*$. The following theorem relates (7.2,50) to Algorithm II.

<u>Theorem 26</u>

The subspaces $\mathcal{L}_i$ satisfy the algorithm

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$$\mathcal{V}^{i+1} = \mathcal{N}[C + DL_i] \cap [A + BL_i]^{-1}\mathcal{V}^i$$

$$\mathcal{V}^0 = \mathcal{Z}$$

for appropriate choices of L_i. These matrices can be obtained from the feedback matrices K_i calculated in Algorithm II.

Proof: Algorithm I (5.2,15) shows that

$$S_i \begin{bmatrix} M_i(Ax + Bu) \\ \\ Cx + Du \end{bmatrix} = \begin{bmatrix} F_{i+1}u + G_{i+1}x \\ \\ M_{i+1}x \end{bmatrix} \qquad (7.2,51)$$

Let $u = L_i x$ where $L_i = -F^+_{i+1}G_{i+1}$. Equation (7.2,51) can be rewritten as

$$S_i \begin{bmatrix} M_i(A + BL_i)x \\ \\ (C + DL_i)x \end{bmatrix} = \begin{bmatrix} 0 \\ \\ M_{i+1}x \end{bmatrix} \qquad (7.2,52)$$

This implies that

(1) $M_{i+1}x = 0$

(2) $(C + DL_i)x = 0$ and $M_i(A + BL_i)x = 0$

Condition (2) can be written as

$$\begin{bmatrix} A + BL_i \\ \\ C + DL_i \end{bmatrix} x = \begin{bmatrix} z \\ \\ 0 \end{bmatrix} \qquad (7.2,53)$$

for some z satisfying $M_i z = 0$. Since x also satisfies condition (1), the

first part of the theorem follows for $\mathcal{V}^i = \mathcal{N}[M_i] = \mathcal{L}_i$.

Algorithm II provides for the explicit representation of the subspaces $\mathcal{L}_i$. It is easily seen that the matrices L_i can be obtained from the feedback matrices K_i by a straight forward calculation. For example, L_1 is calculated as

$$L_1 = W_1K_1 + W_0K_0T_0^{-1}V_1^{-1} \qquad (7.2,54)$$

□

Two other methods for the computation of $\mathcal{L}^*$ have appeared recently. In [53], $\mathcal{L}^*$ is computed by finding compatible sets of eigenvectors in $\mathcal{N}[P(\lambda_i)]$ where P(s) is the system matrix and λ_i is the eigenvalue which corresponds to the eigenvector being computed. In [46] a variation of Algorithm II appears as a numerical procedure applied to the system matrix. The theory is drawn from Kronecker's theory on the structure of pencils of matrices. The presentation there is for system with $D = 0$ and doesn't include the geometric interpretation given here.

7.3. Calculating Invariant Zeros

Algorithm II in Section 7.2 provides a method for calculating canonic form (7.2,42). Once this canonic form is available, the invariant zeros (Definition 5) can easily be calculated from it by using Theorem 14. The invariant zeros are simply the eigenvalues of the submatrix A_7. The invariant zeros can also be easily computed from canonic form (7.2,1). In this case, they are the eigenvalues that are associated with the unreachable modes of (A_4,B_4). This can be seen from (5.2,31); it is also clear from the proof of Theorem 14.

Using Algorithm II, the invariant zeros can be computed for any system (A,B,C,D). In particular, the invariant zeros of a degenerate system can be calculated via Algorithm II. Algorithm II differs in this respect from other computational methods such as the Generalized Eigenvalue Method or the Davison-Wang method. (The latter compute zeros defined by Definition 6). Of course, any procedure whose theoretical basis rests on $\mathcal{L}^*$ (such as the method by inverse systems) will compute the same zeros as does Algorithm II. Algorithm II, however, places no restrictions on the matrices (A,B,C,D) and, therefore, is more general than certain other methods.

The procedure for finding invariant zeros is summarized in Fig. 1 the following flow chart.

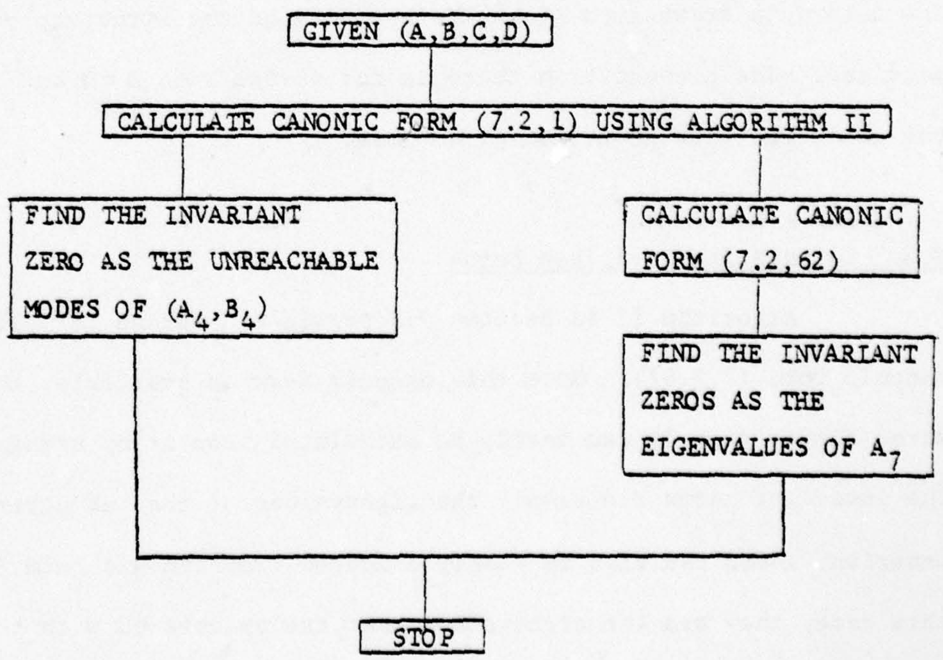

Fig. 1. Calculating Invariant Zeros Via Algorithm II.

Algorithm II is condensed and summarized in the following step by step procedure.

Given an arbitrary system (A,B,C,D).

Step 1

Define $\sigma_0 = 0$, $\delta_0 = 0$, and

$$P_0 = \begin{bmatrix} C & D \\ & \\ A & B \end{bmatrix}$$

Find non-singular input and output space transformations, W_0 and S_0 respectively, such that

$$P_0' = \begin{bmatrix} S_0 & 0 \\ & \\ 0 & I_n \end{bmatrix} P_0 \begin{bmatrix} I_n & 0 \\ & \\ 0 & W_0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \bar{C}^1 & \bar{D}^1 & 0 \\ \tilde{C}^1 & 0 & 0 \\ 0 & 0 & 0 \\ A & B_1^1 & B_2^1 \end{bmatrix} \begin{array}{l} \}r - r_0 \\ \}r_0 \\ \\ \\ \end{array} \\ \begin{array}{ccc} \quad & m_0 & m - m_0 \end{array} \end{array}$$

where $\bar{D}^1$ is a square non-singular matrix and $\tilde{C}^1$ has full row rank.

Step 2

Calculate $\bar{K}_0 = -(\bar{D}^1)^{-1}\bar{C}_1$ and form the $m \times n$ matrix

$$K_0 = \begin{bmatrix} \bar{K}_0 \\ \\ 0 \end{bmatrix}$$

If $r_0 = 0$, set $T_0 = I_n$. Otherwise form the matrix

$$T_0 = \left[\begin{array}{c|c} \multicolumn{2}{c}{\tilde{C}^1} \\ \hline 0 & I_{n-r_0} \end{array}\right]$$

Calculate

$$\begin{bmatrix} I_r & 0 \\ \\ 0 & T_0 \end{bmatrix} P_0' \begin{bmatrix} T_0^{-1} & 0 \\ \\ K_0 & I_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & \bar{D}^1 & 0 \\ I_{r_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{11}^1 & A_{12}^1 & B_{11}^1 & B_{12}^1 \\ A_{21}^1 & A_{22}^1 & B_{21}^1 & B_{22}^1 \end{bmatrix} = P_1$$

If $r_0 = 0$, stop. Otherwise, set $i = 1$ and go to Step 3.

<u>Step 3</u>

Set

$$\begin{bmatrix} x & x & x & x \\ x & A_{12}^i & x & B_{12}^i \\ x & A_{22}^i & x & B_{22}^i \end{bmatrix} = \begin{bmatrix} x & x & x & x \\ x & C^i & x & D^i \\ x & A^i & x & B^i \end{bmatrix} = P_i$$

<u>Step 4</u>

Set $\sigma_i = \sigma_{i-1} + r_i$; $\delta_i = m_i + \delta_{i-1}$.

Find non-singular transformation matrices S_i and $\bar{W}_i$ such that

$$S_i D^i \bar{W}_i = \begin{bmatrix} \bar{D}^{i+1} & 0 \\ \\ 0 & 0 \end{bmatrix} \quad ; \quad S_i C_i = \begin{bmatrix} \bar{C}^{i+1} \\ \tilde{C}^{i+1} \\ 0 \end{bmatrix}$$

where $\bar{D}^{i+1}$ is a square non-singular matrix and $\tilde{C}^{i+1}$ has full row rank.

Form the $n \times n$ non-singular matrix

$$V_i = \begin{bmatrix} I_{\sigma_i} & 0 & 0 \\ 0 & S_i & 0 \\ 0 & 0 & I \end{bmatrix}$$

and the m x m non-singular matrix

$$W_i = \begin{bmatrix} I_{\delta_i} & 0 \\ 0 & \overline{W}_i \end{bmatrix}$$

Calculate

$$P_i' = \begin{bmatrix} I_r & 0 \\ 0 & V_i \end{bmatrix} P_i \begin{bmatrix} V_i^{-1} & 0 \\ 0 & W_i \end{bmatrix} = \begin{array}{l} \begin{bmatrix} x & x & x & x & x \\ x & \overline{C}^{i+1} & x & \overline{D}^{i+1} & 0 \\ x & \tilde{C}^{i+1} & x & 0 & 0 \\ x & 0 & x & 0 & 0 \\ x & A^i & x & B_1^{i+1} & B_2^{i+1} \end{bmatrix} \}r_i \\ \qquad\qquad\qquad\quad m_i \end{array}$$

Step 5

Calculate $\overline{K}_i = -(\overline{D}^{i+1})^{-1}\overline{C}^{i+1}$ and form the m x n matrix

$$K_i = \begin{array}{l} \begin{bmatrix} 0 & 0 \\ 0 & \overline{K}_i \\ 0 & 0 \end{bmatrix} \begin{array}{l} \}\delta_i \\ \\ \\ \end{array} \\ \ \ \sigma_i \end{array}$$

If $r_i = 0$, set $T_i = I_n$. Otherwise, form the n x n matrix

$$T_i = \left[\begin{array}{c|c} I_{\sigma_i} & 0 \\ \hline 0 & \begin{array}{c|c} \multicolumn{2}{c}{\tilde{C}^{i+1}} \\ \hline 0 & I \end{array} \end{array}\right]$$

Calculate

$$P_{i+1} = \begin{bmatrix} I_m & 0 \\ 0 & T_i \end{bmatrix} P_i' \begin{bmatrix} T_i^{-1} & 0 \\ K_i & I_m \end{bmatrix} = \begin{bmatrix} x & x & x & x & x & x \\ x & 0 & 0 & x & \bar{D}^{i+1} & 0 \\ x & I_{r_i} & 0 & x & 0 & 0 \\ x & 0 & 0 & x & 0 & 0 \\ x & A_{11}^{i+1} & A_{12}^{i+1} & x & B_{11}^{i+1} & B_{12}^{i+1} \\ x & A_{21}^{i+1} & A_{22}^{i+1} & x & B_{21}^{i+1} & B_{22}^{i+1} \end{bmatrix}$$

If $r_i = 0$, stop. Otherwise, set $i = i+1$ and go to Step 3. □

The algorithm to further transform the system from (7.2,1) to (7.2,42) is given next. The notation is in terms of a pair (A,B). If this pair is a subsystem of a larger system (as above), the transformation matrices should be embedded in a larger transformation matrix which can be applied to the full system.

Given a pair (A,B); A is $n \times n$ and B is $n \times m$.

Step 1

Define

$$P_0 = [B \quad A]$$

Find a non-singular input space transformation W_0 such that

$$P_0' = P_0 \begin{bmatrix} W_0 \\ I_n \end{bmatrix} = [\underset{\lambda_0}{B'} \quad \underset{m-\lambda_0}{0} \quad A]$$

where B' has full column rank.

Step 2

From the n x n transformation matrix R_0 as

$$R_0 = \begin{bmatrix} B_1' & 0 \\ B_2' & I_{n-\lambda_0} \end{bmatrix}$$

Calculate

$$R_0^{-1} P_0' \begin{bmatrix} I_m \\ R_0 \end{bmatrix} = \begin{bmatrix} I_{\lambda_0} & 0 & A_{11}' & A_{12}' \\ 0 & 0 & \underset{\lambda_0}{A_{21}'} & A_{22}' \end{bmatrix} = P_1$$

Set $\tau_0 = \lambda_0$ and $i = 1$.

Step 3

Set

$$P_i = \begin{bmatrix} x & x & x \\ x & A_{21}^i & A_{22}^i \end{bmatrix} = \begin{bmatrix} x & x & x \\ x & B^i & A^i \end{bmatrix}$$

Step 4

Find a non-singular matrix W_i such that

$$B^i W_i = [\underset{\lambda_i}{B^{i+1}} \quad 0]$$

where B^{i+1} has full row rank. If $\lambda_i = 0$, stop. Otherwise form the n x n

non-singular matrix.

$$U_i = \begin{bmatrix} I & 0 & 0 \\ 0 & W_i & 0 \\ 0 & 0 & I_{n-\tau_{i-1}} \end{bmatrix}$$

Calculate

$$U_i^{-1} P_i \begin{bmatrix} I_m \\ U_i \end{bmatrix} = \begin{bmatrix} x & x & x & x \\ x & B^{i+1} & 0 & A^i \end{bmatrix} = P_i'$$

Step 5

Form the n x n non-singular matrix

$$R_i = \begin{bmatrix} I_{\tau_{i-1}} & 0 & 0 \\ 0 & B_1^{i+1} & 0 \\ 0 & B_2^{i+1} & I \end{bmatrix}$$

Calculate

$$R_i^{-1} P_i' \begin{bmatrix} I_m \\ R_i^{-1} \end{bmatrix} = \begin{bmatrix} x & x & x & x & x \\ x & I_{\lambda_i} & 0 & A_{11}^{i+1} & A_{12}^{i+1} \\ x & 0 & 0 & \underset{\lambda_i}{A_{21}^{i+1}} & A_{22}^{i+1} \end{bmatrix}$$

Set $\tau_i = \tau_{i-1} + \lambda_i$, $i = i+1$ and go to Step 3. □

The following examples illustrate the use of Algorithm II to compute invariant zeros. Note that these examples also illustrate the

geometrical ideas presented in Section 5.2.

Example 17

Consider the system

$$x_{i+1} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 4 & 3 \\ 1 & 0 & 0 & 3 \end{bmatrix} x_i + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u_i$$

$$y_i = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} x_i + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} u_i$$

Calculate the invariant zeros of this system by first finding canonic form (7.2,1) via Algorithm II.

Step 1

Note that if

$$W_0 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

then

$$DW_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is in the required form. Furthermore, $S_0 = I_2$. Then

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$$P_0' = I_6 \left[\begin{array}{cccc|cc} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 3 & 1 & 1 \\ 1 & 0 & 0 & 3 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} I_4 & 0 & 0 \\ \hline 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c|c} 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 3 & 1 & 0 \\ 1 & 0 & 0 & 3 & 0 & 1 \end{array}\right] = \begin{bmatrix} \bar{C}^1 & \bar{D}^1 & 0 \\ \tilde{C}^1 & 0 & 0 \\ A & B_1' & B_2' \end{bmatrix}$$

Note that $r_0 = 1$ and $m_0 = 1$.

<u>Step 2</u>

$$\bar{K}_0 = -(\bar{D}^1)^{-1}\bar{C}^1 = [0 \quad -1 \quad -10]$$

$$K_0 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $r_0 \neq 0$, form the transformation matrix

$$T_0 = \left[\begin{array}{c|ccc} 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \multicolumn{2}{c}{\tilde{C}^1} \\ \hline 0 & I \end{array}\right]$$

$$P_1 = \left[\begin{array}{c|ccc|c|c} 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & -1 & -2 & 2 & 2 & 0 \\ \hline 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc} 0 & 0 & \bar{D}^{1} & 0 \\ I_1 & 0 & 0 & 0 \\ A_{11}^1 & A_{12}^1 & B_{11}^1 & B_{12}^1 \\ A_{21}^1 & A_{22}^1 & B_{21}^1 & B_{22}^1 \end{array}\right]$$

Step 3

Set

$$A_{12}^1 = C^1 = [-1 \ -2 \ 2], \quad B_{12}^1 = D^1 = [0]$$

$$A_{22}^1 = A^1, \quad B_{22}^1 = B^1$$

Step 4

Since D^1 is already in reduced form, it follows that $S_1 = I_1$, $W_1 = I_1$, $P_1 = P_1'$, $C_1 = \tilde{C}_1$, $r_1 = 1$, and $m_1 = 0$.

Step 5

As $D^1 = 0$, $K_1 = 0$. Form the transformation matrix

$$T_1 = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & -2 & 2 \\ \cline{2-4} 0 & 0 & \multicolumn{1}{|c}{1} & 0 \\ 0 & 0 & \multicolumn{1}{|c}{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} I_1 & 0 \\ \hline 0 & \tilde{C}^1 \\ \cline{2-2} & 0 \ \ I_2 \end{array}\right]$$

Then

$$P_2 = \left[\begin{array}{c|c|cc|c|c} x & x & x & x & x & x \\ x & 1 & 0 & 0 & x & 0 \\ \hline x & -1 & -10 & 2 & x & 2 \\ \hline x & +1 & 4 & 0 & x & 0 \\ x & 0 & -1 & 2 & x & 1 \end{array}\right] = \begin{bmatrix} x & x & x & x & x \\ x & I_1 & 0 & x & 0 \\ x & A_{11}^2 & A_{12}^2 & x & B_{12}^2 \\ x & A_{21}^2 & A_{22}^2 & x & B_{22}^2 \end{bmatrix}$$

Cycle 2

Step 3

Identify

$$A_{12}^2 = C^2 = [-10 \quad 2]; \quad B_{12}^2 = D^2 = [2]$$

$$A_{22}^2 = A^2 \quad ; \quad B_{22}^2 = B^2$$

Step 4

Since D^2 has full row rank, $P_2 = P_2'$, $\bar{C}^3 = C^2$, $r_2 = 0$ and $m_2 = 1$.

Step 5

$$\bar{K}_2 = -(\bar{D}^3)^{-1}\bar{C}^3$$

$$= [5 \quad -1]$$

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -1 \end{bmatrix}$$

Since $r_2 = 0$, $T_2 = I_n$. Then

$$P_3 = \left[\begin{array}{c|c|c|c|c} x & x & x & x & x \\ \hline x & 0 & 0 & x & 2 \\ \hline x & 4 & 0 & x & 0 \\ x & 4 & 1 & x & 1 \end{array}\right] = \begin{bmatrix} x & x & x & x \\ x & 0 & x & \bar{D}^3 \\ x & A_{22}^3 & x & B_{21}^3 \end{bmatrix}$$

The algorithm terminates. The system now has the form

$$x_{i+1} = \left[\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 1 & 1 & 4 & 0 \\ 1 & 0 & 4 & 1 \end{array}\right] x_i + \left[\begin{array}{c|c} 2 & 0 \\ \hline -2 & 2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u_i$$

$$y_i = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] x_i + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] u_i$$

Since $B_{22} = 0$, $\mathcal{R}^* = 0$. Therefore, the invariant zeros are just the eigenvalues of A_{22}; i.e. $z = 1$ and $z = 4$, by inspection. □

<u>Example 18</u> [31]

Consider the system

$$x_{i+1} = \begin{bmatrix} -1 & 1 & 3 & -2 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -1 & -5 \end{bmatrix} x_i + \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} u_i$$

$$y_i = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & -1 & -3 & 2 \end{bmatrix} x_i$$

Find the invariant zeros based on the following pairs of inputs and outputs:

a) u_i and y_i

b) u_i^1 and y_i where $u_i = \begin{bmatrix} u_i^1 \\ u_i^2 \end{bmatrix}$

c) u_i and y_i^2 where $y_i = \begin{bmatrix} y_i^1 \\ y_i^2 \end{bmatrix}$

a) u_i and y_i

<u>Step 1</u>

Define

$$P_0 = \begin{bmatrix} C & D \\ A & B \end{bmatrix} = \left[\begin{array}{cccc|cc} 1 & -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & -3 & 2 & 0 & 0 \\ \hline -1 & 1 & 3 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 2 \\ 0 & 1 & -3 & -1 & 1 & 1 \\ 0 & 3 & -1 & -5 & 2 & 2 \end{array}\right]$$

Since $D = 0$, the algorithm is simplified. It is seen that $W_0 = I_2$, $\tilde{C}^1 = C$, $r_0 = 2$, and $m_1 = 0$.

<u>Step 2</u>

Again, $D = 0$ implies $K_0 = 0$. Form the transformation matrix T_0 as

$$T_0 = \left[\begin{array}{c|c} \multicolumn{2}{c}{C} \\ \hline 0 & I_2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & 3 & 0 \\ 0 & -1 & -3 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Then

$$P_1 = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline -1 & -4 & -14 & 4 & 3 & 1 \\ 0 & -4 & -4 & 2 & 0 & -1 \\ \hline 0 & -1 & -6 & 1 & 1 & 1 \\ 0 & -3 & -10 & 1 & 2 & 2 \end{array}\right] = \begin{bmatrix} I_2 & 0 & 0 \\ A_{11}^1 & A_{12}^1 & B_{12}^1 \\ A_{21}^1 & A_{22}^1 & B_{22}^1 \end{bmatrix}$$

Step 3

Identify

$$C^1 = \begin{bmatrix} -14 & 4 \\ -4 & 2 \end{bmatrix} \quad ; \quad D^1 = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} -6. & 1 \\ -10 & 1 \end{bmatrix} \qquad B^1 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Step 4

Note that D^1 has full rank. Therefore, $r_1 = 0$. Hence, $V_1 = I_n$, $W_1 = I_m$ and $P_1 = P_1'$.

Step 5

Now

$$\bar{K}_1 = -(\bar{D}^2)^{-1}\tilde{C}^2$$

$$= - \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -14 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -4 & 2 \end{bmatrix}$$

and

$$K_1 = \begin{bmatrix} 0 & 0 & 6 & -2 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

Since $r_1 = 0$, $T_1 = I_n$. Then

$$P_2 = \left[\begin{array}{c|cc|cc} x & x & & x & \\ \hline & 0 & 0 & 3 & 1 \\ x & & & & \\ & 0 & 0 & 0 & -1 \\ \hline & -6 & 1 & 1 & 1 \\ x & & & & \\ & -10 & 1 & 2 & 2 \end{array}\right] = \begin{bmatrix} x & x & x \\ x & 0 & \bar{D}^2 \\ x & A_{22}^2 & B_{21}^2 \end{bmatrix}$$

The algorithm terminates as $r_1 = 0$. By writing out P_2 and identifying the original system matrices, it follows that

$$x_{i+1} = \left[\begin{array}{cc|cc} -1 & -4 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ \hline 0 & -1 & -4 & 1 \\ 0 & -3 & -6 & 1 \end{array}\right] x_i + \left[\begin{array}{cc} 3 & 1 \\ 0 & -1 \\ \hline 1 & 1 \\ 2 & 2 \end{array}\right] u_i$$

$$y_i = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] x_i$$

This is clearly in canonic form (7.2,1). From P_2, it is seen that $B_{22} = 0$ which implies that $\mathcal{R}^* = 0$. By Theorem 14, the invariant zeros are the eigenvalues of

$$\begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix}$$

These are easily calculated as $z_1 = -1$ and $z_2 = -2$.

b) u_i^1 and y_i

Since the C matrix is unchanged from part (a), P_1 is the same except the last column is deleted:

$$P_1 = \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1 & -4 & -14 & 4 & 3 \\ 0 & -4 & -4 & 2 & 0 \\ \hline 0 & -1 & -6 & 1 & 1 \\ 0 & -3 & -10 & 1 & 2 \end{array}\right]$$

Step 3

Identify

$$C^1 = \begin{bmatrix} -14 & 4 \\ -4 & 2 \end{bmatrix} \quad ; \quad D^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} -6 & 1 \\ -10 & 1 \end{bmatrix} \quad ; \quad B^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Step 4

Note that D^1 is in the required form, so that $S_1 = I_2$, $W_1 = I_1$, and $V_1 = I_n$. Hence $P_1 = P_1'$. Immediately identify

$$\bar{C}^2 = [-14 \quad 4] \quad ; \quad \bar{D}^1 = [3]$$

$$\tilde{C}^2 = [-4 \quad 2]$$

$$P_2 = \left[\begin{array}{c|ccc} x & x & x & x \\ \hline x & 0 & 0 & 3 \\ x & 1 & 0 & 0 \\ x & -1 & 0 & 0 \\ x & 0 & -2 & 2 \end{array}\right] = \left[\begin{array}{c|ccc} x & x & x & x \\ \hline x & 0 & 0 & \bar{D}^2 \\ x & I_1 & 0 & 0 \\ x & A_{11}^2 & A_{12}^2 & B_{11}^2 \\ x & A_{21}^2 & A_{22} & B_{21}^2 \end{array}\right]$$

Cycle 2

Step 3

Identify

$$C^2 = [0] \quad ; \quad D^2 = [0]$$

$$A^2 = [-2] \; ; \quad B^2 = [2]$$

Step 4

Since D^2 is in the required form, it follows that $C^2 = \tilde{C}^3 = [0]$ and $r_2 = 0$.

Step 5

Clearly $\bar{K}_2 = 0$ and $r_2 = 0$ implies $T_2 = I_n$. Hence, the system is in the required form.

$$x_{i+1} = \left[\begin{array}{ccc|c} -1 & -4 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ \hline 0 & -3 & 0 & -2 \end{array}\right] x_i + \left[\begin{array}{c} 3 \\ 0 \\ 0 \\ \hline 2 \end{array}\right] u_i$$

$$y_i = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] x_i$$

From P_2 it is seen that $B_{22}^2 = 0$ so that $\Re^* = 0$. Hence, this system has one invariant zero at $z = -2$. Notice that this system has two unreachable modes but only one of them is associated with an invariant zero.

c) u_i and y_i^2

Let the system be given by

$$x_{i+1} = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 0 & -1 & 3 & -2 \\ 1 & 0 & -3 & -1 \\ 3 & 0 & -1 & -5 \end{bmatrix} x_i + \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} u_i$$

$$y_i = [-1 \quad 0 \quad -3 \quad -2] x_i$$

where the first two state variables have been permuted to ensure the non-singularity of T_0 and the inputs have been permuted to ensure consistency of notation below.

Step 1

Form P_0. Note that $C = \tilde{C}^1$ and $r_0 = 1$

Step 2

Note that $\bar{K}_0 = 0$ as $D = 0$. Form T_0 as before. Then

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$$P_1 = \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline -4 & 0 & -4 & 2 & -1 & 0 \\ \hline -1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -6 & 1 & 1 & 1 \\ -3 & 0 & -10 & 1 & 2 & 2 \end{array}\right] = \begin{bmatrix} I_1 & 0 & 0 \\ A_{11}^1 & A_{12}^1 & B_{12}^1 \\ A_{21}^1 & A_{22}^1 & B_{22}^1 \end{bmatrix}$$

Step 3

Identify

$$C^1 = [0 \;\; -4 \;\; 2] \; ; \; D^1 = [-1 \;\; 0]$$

$$A^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -6 & 1 \\ 0 & -10 & 1 \end{bmatrix} ; \; B^1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Step 4

Since D^1 is in the required form, $P_1' = P_1$. Then

$$C^1 = \bar{C}^2, \;\; \tilde{C}^2 = 0, \;\; r_1 = 0, \;\; m_1 = 1,$$

and

$$B_1^2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \; B_2^2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Step 5

As usual

$$\bar{K}_1 = [0 \;\; -4 \;\; 2]$$

and

$$K_1 = \begin{bmatrix} 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $r_1 = 0$, $T_1 = I_n$, and

$$P_2 = \left[\begin{array}{c|ccc|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline -4 & 0 & 0 & 0 & -1 & 0 \\ \hline -1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -10 & 3 & 1 & 1 \\ -3 & 0 & -18 & 5 & 2 & 2 \end{array}\right] = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ A_{11}^2 & 0 & \bar{D}^2 & 0 \\ A_{12}^2 & A_{22}^2 & B_{21}^2 & B_{22}^2 \end{bmatrix}$$

Since $r_1 = 0$ the algorithm terminates. However, note that $B_{22}^2 \neq 0$ so that $\mathcal{R}^* \neq 0$. Now the invariant zeros could be calculated from the unreachable modes of (A_{22}^2, B_{22}^2). However, here the system will be further reduced to canonic form (7.2,42).

Step 1

Identify

$$P_0 = \left[\begin{array}{c|ccc} 1 & -1 & 0 & 0 \\ 1 & 0 & -10 & 3 \\ 2 & 0 & -18 & 5 \end{array}\right]$$

Immediately, $\lambda_0 = 1$ and $P_0 = P_0'$.

Step 2

$$R_0 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ 2 & 0 & 1 \end{array}\right]$$

Then

$$P_1 = \left[\begin{array}{c|c|cc} 1 & -1 & 0 & 0 \\ \hline 0 & -3 & -10 & 3 \\ 0 & -6 & -18 & 5 \end{array}\right] = \begin{bmatrix} I_1 & A^1_{11} & A^1_{12} \\ 0 & A^1_{21} & A^1_{22} \end{bmatrix}$$

Step 3

Identify

$$A^1_{21} = B^1 = \begin{bmatrix} -3 \\ -6 \end{bmatrix} \; ; \; A^1_{22} = A^1 = \begin{bmatrix} -10 & 3 \\ -18 & 5 \end{bmatrix}$$

Step 4

Since B^1 is in the required form, $\lambda_1 = 1$ and $P_1 = P_1'$.

Step 5

$$R_0 = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & -3 & 0 \\ \cline{3-3} 0 & -6 & 1 \end{array}\right]$$

Then

$$P_2 = \left[\begin{array}{c|c|c|c} 1 & -1 & 0 & 0 \\ \hline 0 & 1 & -4 & -1 \\ \cline{2-4} 0 & 0 & 0 & -1 \end{array}\right] = \begin{bmatrix} x & x & x & x \\ x & I_1 & A^2_{11} & A^2_{12} \\ x & 0 & A^2_{21} & A^2_{22} \end{bmatrix}$$

Cycle 2

Step 3

Identify $A^2_{21} = B_2 = [0]$, $A^2_{22} = A^2 = [-1]$.

<u>Step 4</u>

Since $B_2 = [0]$, $\lambda_2 = 0$ and the algorithm terminates. The full system now has the form

$$x_{i+1} = \left[\begin{array}{c|cc|c} -4 & 0 & 0 & 0 \\ \hline -1 & -1 & 0 & 0 \\ -1 & 1 & -4 & -1 \\ \hline -3 & 0 & 0 & -1 \end{array}\right] x_i + \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{array}\right] u_i$$

$$y_i = [1 \,|\, 0 \quad 0 \quad 0] x_i$$

By inspection there is an invariant zero at $z = -1$. It is interesting to note that the subsystem

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -4 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has two modes with eigenvalues of -1 but with linearly independent eigenvectors. One is reachable and one is not.

CHAPTER 8

CONCLUSION

This work has three major areas of emphasis. The first is a survey of selected literature on zeros for linear time-invariant multivariable systems. The second is a survey of algorithms for calculating zeros. This includes the properties of zeros on which these algorithms are based. The third area of emphasis is the presentation of a new algorithm for the calculation of invariant zeros.

Transmission zeros were defined for a transfer function matrix using the Smith-McMillan form. The motivation for this definition comes from the fact that these transmission zeros are frequencies whose transmission through the system is blocked; a generalization of zeros of a scalar transfer function. These transmission zeros have other properties (i.e. alternative definitions) which can be thought of as generalizations of properties of zeros of a scalar transfer function. For example, the transmission zeros are the poles of an inverse system (if it exists). All of these properties are closely interrelated through the Smith-McMillan form.

A similar analysis is carried out for zeros defined for state space systems by using the system matrix and the Smith form of the system matrix. In this way invariant zeros are introduced. The transmission zeros are contained in the set of invariant zeros; however, the analysis is complicated by the appearance of decoupling zeros. The decoupling zeros turn out to be just the eigenvalues associated with the uncontrollable

and/or unobservable modes of the system. The invariant zeros, in general, contain all of the transmission zeros but may not contain all of the decoupling zeros. A final set of zeros is introduced, called system zeros. This set contains all of the transmission zeros and decoupling zeros. The invariant zeros are a (sometimes proper) subset of system zeros.

As with transmission zeros, invariant zeros have several properties which can be considered generalizations of properties of zeros defined for a scalar transfer function. In fact, the motivation for defining invariant zeros is, again, to identify those frequencies whose propagation through the system is blocked. In addition, the invariant zeros are shown to be unaffected by state feedback, to be the limiting positions of the system poles under high gain feedback, and to be contained in the set of poles of an inverse system (when it exists). System zeros (and invariant zeros) are also unaffected by input, output, and state space transformations as well as output feedback.

The geometrical properties of zeros are also discussed in detail. It is shown that invariant zeros are related to $\mathcal{L}^*$, the maximal null output (A,B)-invariant subspace and to $\mathcal{R}^*$, the maximal null output reachability subspaces. These two subspaces and invariant zeros play a key role in the discussion of inverse systems. They also provide the theoretical basis for a new algorithm for computing invariant zeros.

It turns out that the definitions of the various zeros are not very convenient for actually computing them, either by hand or by digital computer. These properties, then, provide the basis for several algorithms for calculating zeros. Sometimes these properties can be applied directly.

For instance, for a certain class of systems, the inverse system can easily be computed. High gain feedback can also be applied directly. This property has also lead to a number of other algorithms including the NAM algorithm. Finally, invariant zeros have been computed using a generalized eigenvector method in combination with a QZ-algorithm. This method has also been cast in the more general setting of computing the Kronecker structure of a pencil of matrices.

In Chapter 7, a new algorithm for the calculation of invariant zeros is introduced. This algorithm is a sequence of transformations on the system. Then a feedback matrix can easily be identified which will place the system in a canonic form such that the subspaces $\mathfrak{L}^*$ and $\mathfrak{R}^*$ are displayed explicitly. From this canonic form, the invariant zeros can easily be calculated. The fact that this algorithm actually calculates $\mathfrak{L}^*$ and $\mathfrak{R}^*$ suggests that its use is not restricted just calculating invariant zeros. In fact, it is useful in carrying out the construction of inverse systems as presented in Chapter 5. This will be explored in a future paper.

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